São Paulo Journal of Mathematical Sciences **5**, 2 (2011), 233–248

Inverse Problem of Variational Calculus

Pedro Gonçalves Henriques

Centro de Análise Matemática, Geometria e Sistemas Dinâmicos
Departamento de Matemática, Instituto Superior Técnico
Av. Rovisco Pais, 1049-001 Lisboa, Portugal

E-mail address: phenr@math.ist.utl.pt

Abstract. We will discuss the so-called mixed endpoint conditions for variational problems with non-holonomic constraints given by form actions of order greater than one. We will present some results and discuss the inverse problem of Calculus of Variations.

1. **Introduction**

In the present text we describe new results for the the inverse problem of Variational Calculus for multiple integrals in the context of exterior differential systems. We deal with non-holonomic constraints in the setting of the mixed endpoint conditions defined by P. A. Griffiths ([16]). This work is a follow up of ([29]). In section I and II we present a short review of the latter work. In section III we discuss the inverse problem of Variational Calculus, and conclude in section IV with a study of the Generalized Lagrange Problem with non-holonomic constraints.

Caratheodory [1929], Weil-De Donder [1936] and Lepage [1936-1942] were the first to study Variational Calculus for multiple integrals. Later Dedecker [1953-1977], Liesen [1967], R. Hermann [1966], H. Goldschmidt and S. Sternberg [1973], Ouzilou [1972], D. Krupka [1970-1975], I. M. Anderson [1980], P. L. Garcia and A. Pérez-Rendón [1969-1978], C. Günther [1987], Edelen [1961] and Rund [1966], gave their contributions to this field.

In 1983 Griffiths (see [16]) presented a new approach to variational problems in the context of exterior differential systems. His work in the set of one-dimensional integral manifolds of a differential system (I^*, L^*) addressed the problem of finding extrema for the functional ϕ using intrinsic

Partially supported by FCT/Portugal through the Center for Mathematical Analysis, Geometry, and Dynamical Systems.

entities. He established the mixed endpoint conditions for problems with non-holonomic constraints.

The Inverse Problem of Variational Calculus was presented in 1887 by Helmholtz in the following way: given $P_i = P_i(x, u^j, u^j_x, u^j_{xx})$ is there a Lagrangian $L(x, u^j, u^j_x)$ such that $E_i(L) = \partial L/\partial u^i - D_x \partial L/\partial u^i_x = P_i$ where $D_x = \partial/\partial x + u^i_x \partial/\partial u^i + u^i_{xx}\partial/\partial u^i_x$? He found necessary conditions for P_i to be a Euler-Lagrange system (see (3.1) (3.2) and (3.3)). Years later these equations where proved to be locally sufficient conditions.

I. M. Anderson [1992], [1980], P. J. Olver [1986], F. Takens [1979], W. M. Tulczyjew [1980] and A. M. Vinagradov [1964] generalized Helmholtz's conditions both to higher order systems of partial differential equations and to multiple integrals.

1.1. **Integral manifolds and valued differential systems.** Let X be a manifold and assume that a Pfaffian differential system (I^*, L^*) is given by:

i) a subbundle $I^* \subset T^*X$,
ii) another subbundle $L^* \subset T^*X$ with $I^* \subset L^* \subset T^*X$,

such that the rank $(L^*/I^*) = n$ (with n being a natural number).

An integral manifold of (I^*, L^*) is given by an oriented connected compact n-dimensional smooth manifold N (possibly with a piecewise smooth boundary N) together with a smooth mapping

$$f : N \to X$$

satisfying:

$$ {I^*_{f(x)}}^{\perp} = {L^*_{f(x)}}^{\perp} + f_*(TN), \tag{1.1}$$

for all $x \in N$.

$V(I^*, L^*)$ is the collection of integral manifolds f of (I^*, L^*).

A valued differential system is a triple (I^*, L^*, φ), where (I^*, L^*) is a Pfaffian differential system and φ is an n-form on X.

We define the functional ϕ associated with (I^*, L^*, φ) in $V(I^*, L^*)$ by:

$$\phi : V(I^*, L^*) \to R,$$

$$f \to \phi[f] = \int f^*\varphi. \tag{1.2}$$

1.2. **Local embeddability.** Let us assume that

$$d(C^\infty(X, L^*)) \subset C^\infty(X, L^* \wedge T^*X)$$

and let $d' = \dim X$, $\quad s = \mathrm{rank} I^*$, $(d(C^\infty(X, L^*))$ is the set of images by the exterior derivative of $C^\infty(X, L^*))$. Using the Frobenius theorem, we can set for every $p \in X$ a chart coordinate system $\{u^1, ..., u^{s+n}, v^1, ..., v^{d'-s-n}\}$ so that:

i)

$$L^* = \text{ span}\{du^\alpha | 1 \leq \alpha \leq s+n\}) \tag{1.3}$$

ii)

$$L^{*\perp} = \text{ span}\{\frac{\partial}{\partial v^i} | 1 \leq i \leq d' - s - n\} \tag{1.4}$$

for an open subset U of X with $p \in U$.

Definition 1.1. *Let (I^*, L^*) be a Pfaffian differential system with*

$$d(C^\infty(X, L)) \subset C^\infty(X, L^* \wedge T^*X) \ .$$

(I^*, L^*) is locally embeddable if for every $p \in X$ there exists an open neighborhood U of p and local coframes $CF = \{\theta_1, ..., \theta_s\}$ for I^* and $CF' = \{\theta_1, ..., \theta_s, du"^{s+1}, du"^{s+n}\}$ for L^*_U satisfying the following conditions:

(i) $\delta(I^*_U \wedge \Omega) \subset T^*U \wedge \wedge^n(L^*_U)/T^*U \wedge I^*_U \wedge \wedge^{n-1}(L^*))$
(ii) *Ker δ is a constant rank subbundle of $I^* \wedge \Omega$,*

where $\Omega = \text{span}\{du"^{s+1} \wedge ... \wedge \widehat{du"^{s+\beta}} \wedge ... \wedge du"^{s+n}\}$, $\widehat{du"^{s+\beta}}$ - means deletion of $s+b$ factor (for $n = 1, \widehat{du"^{s+1}} = 1$).

The map $\delta : I^* \wedge \Omega \to \wedge^{n+1}(T^*U)/I^*_u \wedge (\wedge^n(T^*U))$ is induced by:

$$d : C^\infty(U, I^* \wedge \Omega) \to C^\infty(U, \wedge^{n+1}(T^*U))$$

in $I^* \wedge \Omega$, i.e, if I^* has no Cauchy characteristics, the structure equations are locally:

$$d\theta^i \equiv \pi^i_j \wedge du"^{s+j} + A^{ij'}_{i'\alpha}\pi^{i'}_{j'} \wedge \theta^\alpha + B^i_{\alpha\beta}\theta^\alpha \wedge du"^{s+\beta} mod I \wedge I \tag{1.5}$$

$1 \leq i, i', \alpha \leq s, 1 \leq j, j', \beta \leq n, I = C^\infty(X, I^*)$.

In ([26]) we showed that if (I^*, L^*) is locally embeddable, there exist locally defined mappings that induce (I^*, L^*) from the canonical system in $J^1(R^n, R^s)$ with possible constraints, giving a local correspondence between these two differential systems. These sets of systems form a general class that can be described locally by the canonical system of $J^1(R^n, R^s)$.

1.3. **The Cartan system.** Let (I^*, L^*, φ) be a valued differential system on X, and W be the total space of I^*. Let χ be the canonical form on T^*X, and i the inclusion map $W \overset{i}{\hookrightarrow} T^*X$.

Let us assume that there exists a local n-form ω inducing a nonzero section of $\wedge^n(L^*/I^*)$ with the following form:

$$\omega = \omega^1 \wedge ... \wedge \omega^n \tag{1.6}$$

$$\omega_i = (-1)^{i-1}\omega^1 \wedge ... \wedge \widehat{\omega^i} ... \wedge \omega^n \tag{1.7}$$

Let W^n be the n-Cartesian power of W, and Z be a subset of W^n defined by $Z = \{z \in W^n : \pi'(z) \in \Delta X^n\}$, where π' is the natural projection $\pi'(z) : W^n \to X$, and ΔX^n is the diagonal submanifold of X^n.

The subset Z is a vector subbundle over X and $dimZ = d + sn$ with $d \leq d'$. We define

$$\Psi = d\psi \tag{1.8}$$

where ψ is given by

$$\psi = \pi^*\varphi + (\pi^j oi')^*[i^*(\chi)] \wedge \pi^*\omega_j. \tag{1.9}$$

π^j is the natural projection into the j^{th} component $\pi^j : W^n \to W$, i' is the inclusion map $Z \to W^n$ and π is the natural projection $\pi : Z \to X$. Locally $(\pi^j oi')^*[i^*(\chi)] \wedge \pi^*\omega_j = \lambda^i_j \theta_i{}^j$ with $\theta_i{}^j = \theta^j \wedge \omega_j$.

Definition 1.2. *Given the* $n+1$*-form* Ψ*, the Cartan system* $C(\Psi)$ *is the ideal generated by the set of* n*-forms* $\{v \lrcorner \Psi \quad where \quad v \in C^\infty(Z, TZ)\}$*. An integral manifold of* $(C(\Psi), \omega)$ *is given by an oriented connected compact* n*-dimensional smooth manifold* N *(possibly with a piecewise smooth boundary* ∂N*) together with a smooth mapping* $f : N \to X$ *satisfying:*

$$f^*\theta = 0 \quad for\ every \quad \theta \in C(\Psi) \tag{1.10}$$

and

$$f^*(\omega) \neq 0. \tag{1.11}$$

Solutions of $(C(\Psi), \omega)$ projected in X will give a candidate for extremum of ϕ with suitable boundary conditions.

$$\delta\phi = \int_{f(N)} v \lrcorner d\psi + d(v \lrcorner \psi) \tag{1.12}$$

1.4. **The momentum space.** Suppose we are given on Z:

(i) a closed $(n+1)$-form Ψ with the associated Cartan system $C(\Psi)$,
(ii) $\pi'^*\omega$: the pull back to Z of ω, the n-form which induces a nonzero section on $\wedge^n(L^*/I^*)$.

Definition 1.3. *Let $(C(\Psi),\pi'^*\omega)_n$ be the ideal generated by $(C(\Psi),\pi'^*\omega)$ in $C^\infty(Z,\wedge^n T^*Z)$, $z_0\in Z$ and E_0^p a p-dimensional subspace of the tangent space T_{z_0}. We say that $[z_0,E_0^p]$ is a p-dimensional integral element of $(C(\Psi),\pi'^*\omega)_n$ if*

(i) $<E_0^p,\alpha>=0$ *for all* $(C(\Psi),\pi'^*\omega)_n$
(ii) $<E_0^p\lrcorner\omega>\neq 0$

The set of integral elements $[z_0,E_0^n]$ gives a subset

$$V_n(C(\Psi),\pi^*\omega))\subset G_n(Z).$$

Denoting by π" the projection $G_n(Z)\to Z$ and assuming regularity at each step, one inductively defines:

$$\begin{aligned} Z_1=\pi"(V_n(C(\Psi),\pi^*\omega)),V_n'(C(\Psi),\pi^*\omega)))=\\ \{E\in V_n(C(\Psi),\pi^*\omega):E \text{ tangent to } Z_1\} \end{aligned}\tag{1.13}$$

$$\begin{aligned} Z_2=\pi"(V_n'(C(\Psi),\pi^*\omega)),V"_n(C(\Psi),\pi^*\omega)))=\\ \{E\in V_n'(C(\Psi,\pi^*\omega):E \text{ tangent to } Z_2\}... \end{aligned}\tag{1.14}$$

Definition 1.4. *(I^*,L^*,φ) is a locally embeddable valued differential system, and $\omega=\omega^1\wedge...\wedge\omega^n$. If there exists a $k_0\in N$, such that $Z_{k_0}=Z_{k_0+1}=...=Z_{k_0+n'}(n'\in N)$ in the above construction, with:*

(i) *Z_{k_0} being a manifold of dimension $(n+1)m+n$ for $m\in N$, and*
(ii) *$(C(\Psi),\pi^*\omega)_{Z_{k_0}}$ being a differential system in Z_{k_0} with $r_n=0$ (Cartan number in Cartan-Kähler Theorem) for all $V_{n-1}(C(\Psi),\pi^*\omega))$.*

Then (I^*,L^*,φ) is a non-degenerate valued differential system, and $Z=Y$ is called the momentum space.

For $n=1$ we follow [16] and replace condition (ii) by $\psi\wedge\Psi^n\neq 0$ on Z_{k_0}.

We call $(C(\Psi),\pi^*\omega)_Y$ the prolongation of $(C(\Psi),\pi^*\omega)$ in the momentum space. By construction, the differential system $(C(\Psi),\pi^*\omega)_Y$ satisfies:

(i) the projection $(C(\Psi),\pi^*\omega)\to Y$ is surjective,
(ii) the integral manifolds of $(C(\Psi),\pi'^*\omega)$ on Z coincide with those of $(C(\Psi),\pi^*\omega)$ on Y.

1.5. **Well-posed valued differential systems.** Suppose we have the following subbundles of T^*X such that:

(a)

$$\begin{array}{ccccc} I^* & \subset & L^* & \subset & T^*X \\ \cup & & \cup & & \\ P^* & \subset & M^* & & \end{array} \tag{1.15}$$

(b) the locally given n-form ω also induces a nonzero section on $\wedge^n(M^*/P^*)$,

(c) $Y \subset (P^*)^n|_{\Delta X^n}$, with Y a subbundle of $(P^*)^n|_{\Delta X^n}$.

Definition 1.5. $(I^*, L^*, \varphi, P^*, M^*)$ *is a well-posed valued differential system if the following conditions are satisfied:*

(i) (I^*, L^*, φ) *is a non-degenerate valued differential system (with* $dimY = (n+1)m + n$*) and* $\varphi = L\omega$ *for a smooth function* L *on* X*;*

(ii) *there exists a subbundle* P^* *of* I^* *of rank* m *and a subbundle* M^* *of* L^* *of rank* $m+n$ *as in (1.15);*

(iii) $\pi"^*M^* =$*span*$\{\pi^*\theta | \theta \in C^\infty(X, M^*)\}$ *is completely integrable on* Y*, where* $\pi" = \pi \circ i$*, and* i *once more denotes the inclusion mapping* $Y \to Z$ *and* π *the projection* $Z \to X$*.*

$CF = \{\theta^\alpha, du^{s+j}, \pi^{i'}_{j'}, \pi^{i"}_{j} | 1 \leq \alpha \leq s, 1 \leq i' \leq s_l, j' \in L_{i'}, s_{l+1} \leq i" \leq s, 1 \leq j \leq n\}$ for T^*X with $L_{i'} \subset \{k \in N, 1 \leq k \leq n\}$:

(i)

$$I^* = \text{ span}\{\theta^\alpha | 1 \leq \alpha \leq s\}; \tag{1.16}$$

(ii)

$$L^* = \text{ span}\{\theta^\alpha, du^{s+j} | 1 \leq \alpha \leq s, 1 \leq j \leq n\}; \tag{1.17}$$

(iii) $T^*X = L^* \oplus R^*$ ($\oplus$ denotes a direct sum) with $R^* = \text{span}\{\pi^{i'}_{j'}, \pi^{i"}_{j} | 1 \leq i' \leq s_l, j' \in L_{i'}, s_{l+1} \leq i" \leq s, 1 \leq j \leq n\}$;

(iv)

$$d\theta^{i'}_{j"} \equiv 0 \text{ mod } I, \text{ for } j" \notin L_{i'}\{\theta^{i'}_{j"} = \theta^{i'} \wedge \omega_{j"}\}; \tag{1.18}$$

(v)

$$d\theta^{i'}_{j'} \equiv \pi^{i'}_{j'} \wedge \omega \text{ mod } I, \text{ for } j' \in L_{i'}; \tag{1.19}$$

(vi)

$$d\theta^{i"}_{j} \equiv \pi^{i"}_{j} \wedge \omega \text{ mod } I, \text{ when } 1 \leq j \leq n; \tag{1.20}$$

(vii) $\pi^{i'}_{j'}, \pi^{i"}_{j}$ are linearly independent mod L.

In [29] we presented a set of boundary conditions for different types of Well-posed valued differential systems. Solutions of the Cartan system with these boundary conditions have null first variations i.e. they are soutions of the Euler-Lagrange system.

2. **Generalized Lagrange Problem.**

Let $X = J^1(R^n, R^m)$ (the 1 jet manifold), with the canonical system I^* defined on X (i.e. $I^* = \text{span}\{\theta^\alpha = dy^\alpha - y^\alpha_{x^i} dx^i\}$). Let $\varphi = L\omega$ with $\omega = dx^1 \wedge ... \wedge dx^n$. We choose $x^1, ..., x^n$ to be coordinates for R^n, and $y^1, ..., y^m$ coordinates for R^m.

Definition 2.1. *Let f be a solution to the canonical differential system I^*, with the independence condition given by $L^* = span\{I^*, dx^1, ..., dx^n\}$ and $k \in N$. The family $F(x, t_1, ..., t_k)$ of integral manifolds of (I^*, L^*) is a k-parameter variation of f if:*

(i) *$F(x, t_1, ..., t_k)$ is smooth with $(t_1, ..., t_k) \in [0, \epsilon_1] \times ... \times [0, \epsilon_k]$, for $\epsilon_i > 0, 1 \leq i \leq k$,*
(ii) *$F_{(t_1,...,t_k)} \doteq F(x, t_1, ..., t_k) \in V(I^*, L^*)$ for all $(t_1, ..., t_k) \in [0, \epsilon_1] \times ... \times [0, \epsilon_k]$,*
(iii) *$F(x, 0) = f(x)$ for all $x \in N, N \subset R$.*

$F_*(\frac{\partial}{\partial t_i})$ is an infinitesimal variation of F.

We will consider variations satisfying the condition $\pi"(F(x,t)) = \pi"(f(x))$ for all $x \in \partial N$ and $t \in [0, \epsilon]$ ($\pi"$ is the projection $J^1(R^n, R^m) \to R^n$).

Without loss of generality we can choose v so that $v \lrcorner dx^i = 0$, thus replacing a one parameter variation of f by another that has the same first and second variation while satisfying:

$$\pi"(F(x,t))_N = id_N \tag{2.1}$$

for all $t \in [0, \epsilon]$).

The framework of this Lagrange problem subjected to constrains represents a very important set of problems for the study of Calculus of Variations.

3. **Inverse problem for calculus of variations.**

3.1. **Two Examples.** In 1887, Helmholtz solved the following problem:

Example 1. *Given $P_i = P_i(x, u^j, u^j_x, u^j_{xx})$. Is there a Lagrangian $L(x, u^j, u^j_x)$ such that $E_i(L) = \partial L/\partial u^i - D_x \partial L/\partial u^i_x = P_i$ where $D_x = \partial/\partial x + u^i_x \partial/\partial u^i + u^i_{xx} \partial/\partial u^i_x$? He found the necessary conditions for P_i :*

(i)
$$\partial P_i/\partial u^j_{xx} = \partial P_j/\partial u^i_{xx}, \tag{3.1}$$
(ii)
$$\partial P_i/\partial u^j_x = \partial P_j/\partial u^i_x + 2D_x\partial P_j/\partial u^i_{xx}, \tag{3.2}$$
(iii)
$$\partial P_i/\partial u^j = \partial P_j/\partial u^i - D_x\partial P_j/\partial u^i_x + D_{xx}\partial P_j/\partial u^i_{xx}. \tag{3.3}$$

This problem led to the following studies [4].

(i) - the derivation and analysis of Helmholtz conditions as necessary and (locally) sufficient conditions for a differential operator to coincide with the Euler-Lagrange operator for some Lagrangian;
ii) - the characterization of the obstructions to the existence of global variational principles for different operators defined on manifolds;
iii) - the invariant inverse problem for different operators with symmetry; and
(iv) - the variational multiplier problem wherein variational principles are found, not for a given differential operator, but rather for the differential equations determined by that operator.

That is: find a matrix $B = [B^j_i]$ such that $B^j_i P_j = E_i(L)$ for some L, with B being non-singular.

Let $E \to M$ be a fibered manifold. $J^\infty(E)$ is the infinite jet of E.

Let
$$\theta^i = du^i - u^i_x dx \tag{3.4}$$
$$\theta^i_x = du^i_x - u^i_{xx} dx \tag{3.5}$$
and
$$\begin{aligned}\Omega_P = P_i\theta^i \wedge dx &+ 1/2[\partial P_i/\partial u^i_x - D_x\partial P_i/\partial u^i_{xx}]\theta^i \wedge \theta^j \\ &+1/2[\partial P_i/\partial u^i_{xx} + \partial P_j/\partial u^i_{xx}]\theta^i \wedge \theta^j_x.\end{aligned} \tag{3.6}$$

If P satisfies the Helmholtz conditions, then $d\Omega_P = 0$. If Ω_P is exact (equivalently if $H^{n+1}(E)$ n+1 de Rham cohomology group of E is trivial), then P_i is globally variational.

If $\theta_L = Ldx + \partial L/\partial u^i_x\theta^i$, then $d\theta_L = \Omega_P$.

In 1913, Volterra [50] showed that if $L = \int_N u^i P_i(x, tu^j, tu^j_x, tu^j_{xx})dt$ where $N = [0, 1]$, then:
$$E_i(L) = P_i. \tag{3.7}$$

We have a global solution to the inverse problem in the case of one independent variable and to equations $P_i = 0$ of second order.

Vaingberg 1964 [49] generalized this result, however this Lagrangian is usually of much higher order than necessary.

From [5] one can derive the following theorem:

Theorem 3.1. *Let Δ be a differential operator of order $2k$*

$$\Delta = P_\beta(x^i, u^j, u^j_1, ..., u^j_{2k})\theta^\beta \wedge \omega. \tag{3.8}$$

Then Δ is the Euler-Lagrange operator of a $k-th$ order Lagrangian $L = L(x^i, u^j, u^j_1, ..., u^j_k)$, if and only if Δ satifies the higher order Helmholtz conditions, and the functions

$$p_\beta(t) = P_\beta(x^i, u^j, u^j_1, ..., u^j_k, tu^j_{k+1} ..., t^k u^j_{2k}) \tag{3.9}$$

are polynomials in t of degree less or equal to k.

The functions u^j_k denote all possible k^{th}-order derivatives of u^j, $1 \leq \beta, j \leq m$ and $1 \leq i \leq n$, $\theta^\beta = dy^\beta - y^\beta_{x^i} dx^i$ and $\omega = dx^1 \wedge ... \wedge dx^n$.

Example 2. *Let $T = T(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{xy}, , ..., u_{zz})$ be a second order operator. We assume that T is a smooth function.*
Let $L = L(x, y, z, u, u_x, u_y, u_z)$ be a first-order operator with L being a smooth function.

$$E[L] = \partial L/\partial u - D_x \partial L/\partial u_x - D_y \partial L/\partial u_y - D_z \partial L/\partial u_z$$

where

$$D_x = \partial/\partial x + u_x \partial/\partial u + u_{xx} \partial/\partial u_x + u_{xy} \partial/\partial u_y + ...$$

Let v be a lift to the momentum space of an infinitesimal variation $F_*(\partial/\partial t)$ of $f = \pi o g$, where g is a solution of $(C(\Psi), \pi^*\omega)$. The Lie-derivative of $\psi = \pi^*\varphi + (\pi^j o i')^*[i^*(\chi)] \wedge \pi^*\omega_j$ along v is:

$$\begin{aligned} v \lrcorner d\psi + d(v \lrcorner \psi) = E[L](u)v^1\pi^*(dx \wedge dy \wedge dz) \\ +d(\partial L/\partial u_x v^1 \pi^*(dy \wedge dz) - \partial L/\partial u_y v^1 \pi^*(dx \wedge dz) \\ +\partial L/\partial u_z v^1 \pi^*(dx \wedge dy)) \end{aligned} \tag{3.10}$$

Suppose that for every vector v with $\pi_* v \in T_f V(I^*, L^*, \varphi, [h])$ we have a vector w with $\pi_* w \in T_f V(I^*, L^*, \varphi, [h])$ such that

$$\begin{aligned} v \lrcorner d\psi + d(v \lrcorner \psi) = T[u]v^1\pi^*(dx \wedge dy \wedge dz) \\ +d(\partial L/\partial u_x w^1 \pi^*(dy \wedge dz) - \partial L/\partial u_y w^1 \pi^*(dx \wedge dz) \\ +\partial L/\partial u_z w^1 \pi^*(dx \wedge dy)). \end{aligned} \tag{3.11}$$

Then we have $T[u] = E[L](u)$.

If we identify e_1 with $\pi^*(dy \wedge dz)$, e_2 with $\pi^*(dz \wedge dy)$ and e_3 with $\pi^*(dx \wedge dy)$ at each point of the integral manifold of $(C(\Psi), \pi^*\omega)$, we can write:

$$d(\partial L/\partial u_x v^1 \pi^*(dy \wedge dz) - \partial L/\partial u_y v^1 \pi^*(dx \wedge dz) + \partial L/\partial u_z v^1 \pi^*(dx \wedge dy))$$

$$= DivV[u]\pi^*(dx \wedge dy \wedge dz),$$

where

$$V[u] = \partial L/\partial u_x v^1 e_1 + \partial L/\partial u_y v^1 e_2 + \partial L/\partial u_z v^1 e_3. \tag{3.12}$$

We can conclude that

$$v \lrcorner d\psi + d(v \lrcorner \psi) = (E[L](u)v + DivV[u])\pi^*(dx \wedge dy \wedge dz).$$

We have

$$E[L](u) = 0 \quad \text{if} \quad L[u] = DivW[u]. \tag{3.13}$$

Suppose $T[u] = E[L](u)$. Then

$$E(E[L](u)v + DivV[u]) = E[T(u)v].$$

Let

$$\psi' = \pi^* T\omega + (\pi^j oi')^*[i * (\chi)]\pi^*\omega_j \tag{3.14}$$

If we define

$$H(T)[v]\pi^*(dx \wedge dy \wedge dz) = v \lrcorner d\psi' + d(v \lrcorner \psi') - E[T(u)v]\pi^*(dx \wedge dy \wedge dz),$$

then $H(T) = 0$ if $T[u]$ is Euler-Lagrange. Helmholtz equations are:

(i) $\partial T/\partial u_x = D_x \partial T/\partial u_{xx} + 1/2 D_y \partial T/\partial u_{xy} + 1/2 D_z \partial T/\partial u_{xz}$,
(ii) $\partial T/\partial u_y = D_y \partial T/\partial u_{xx} + 1/2 D_x \partial T/\partial u_{yx} + 1/2 D_z \partial T/\partial u_{yz}$,
(iii) $\partial T/\partial u_z = D_z \partial T/\partial u_{xx} + 1/2 D_x \partial T/\partial u_{zx} + 1/2 D_y \partial T/\partial uzz$.

We have a sequence of spaces

$$0 \to R \to F[u] \xrightarrow{Grad} V(u) \xrightarrow{Curl} V(u) \xrightarrow{Div} F(u) \xrightarrow{E} F(u) \xrightarrow{H} V(u) \tag{3.15}$$

that is a cochain complex, the Euler-Lagrange complex, where $F[u]$ is the set of smooth functions $F(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{xy}, , ..., u_{zz})$, $V[u]$ is the set of vector fields defined in R^n with $F[u]$ coeficients. This complex is exact and thus the inverse problem is solved in this second example.

3.2. **Variational Bicomplex.** Let us introduce now a very important tool for a globalization of the inverse problem [40].

Definition 3.1. *A p form ω on $J^\infty(E)$ is said to be of type (r, s), where $r + s = p$, if at each point x of $J^\infty(E)$,*

$$\omega(X_1, X_2, ..., X_p) = 0, \tag{3.16}$$

whenever either

(i) *more than s of the vectors $X_1, X_2, ..., X_p$ are π_M^∞ vertical, or*
(ii) *more than r of the vectors $X_1, X_2, ..., X_p$ annihilate all contact one forms.*

Note: $\Omega^{r,s}$ denotes the space of type (r, s) forms on $J^\infty(E)$.

(i) $\pi : E \to M$ is a fiber bundle.
(ii) In some cases we can assume that there exists a transformation group G acting on E.
(iii) There exists a set of differential equations on sections of E.

$$d = d_H + d_V$$

$$d_H : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r+1,s}(J^\infty(E)), \tag{3.17}$$

$$d_V : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r,s+1}(J^\infty(E)) \tag{3.18}$$

$$d_H^2 = 0, d_H d_V = -d_V d_H, d_V^2 = 0 \tag{3.19}$$

In local coordinates

$$d_H f = [\partial f/\partial x^i + u\alpha_i \partial f/\partial u^\alpha + u^\alpha_{ij}\partial f/\partial u^\alpha_j + ...]dx^i \tag{3.20}$$

$$d_V f = \partial f/\partial u^\alpha \theta^\alpha + \partial f/\partial u^\alpha_i \theta^\alpha_i + ... \tag{3.21}$$

I is locally given by:

$$I = \Omega^{r,s}(J^\infty(E))) \to \Omega^{r,s}(J^\infty(E))), \tag{3.22}$$

$$I(\omega) = \frac{1}{s}\theta^\alpha \wedge [(\partial/\partial u^\alpha \lrcorner \omega) - D_i((\partial/\partial u^\alpha_i \lrcorner \omega) + D_{ij}((\partial/\partial u^\alpha_{ij} \lrcorner \omega) - ...] \tag{3.23}$$

Definition 3.2. *The sequences of spaces*

$$\begin{array}{cccccccccc}
 & \uparrow d_V & & & & & \uparrow d_V & \uparrow \delta_V & \\
0 & \to \Omega^{0,3} & & & \dots & & \to \Omega^{n,3} & \xrightarrow{I} F^3 & \to 0 \\
 & \uparrow d_V & \uparrow d_V & \uparrow d_V & \dots & \uparrow d_V & \uparrow d_V & \uparrow \delta_V & \\
0 & \to \Omega^{0,2} & \xrightarrow{d_H} \Omega^{1,2} & \xrightarrow{d_H} \Omega^{2,2} & \dots & \xrightarrow{d_H} \Omega^{n-1,2} & \xrightarrow{d_H} \Omega^{n,2} & \xrightarrow{I} F^2 & \to 0 \\
 & \uparrow d_V & \uparrow d_V & \uparrow d_V & \dots & \uparrow d_V & \uparrow d_V & \uparrow \delta_V & \\
0 & \to \Omega^{0,1} & \xrightarrow{d_H} \Omega^{1,1} & \xrightarrow{d_H} \Omega^{2,1} & \dots & \xrightarrow{d_H} \Omega^{n-1,1} & \xrightarrow{d_H} \Omega^{n,1} & \xrightarrow{I} F^1 & \to 0 \\
 & \uparrow d_V & \uparrow d_V & \uparrow d_V & \dots & \uparrow d_V & \uparrow d_V & \uparrow \delta_V & \\
0 \to R & \to \Omega^{0,0} & \xrightarrow{d_H} \Omega^{1,0} & \xrightarrow{d_H} \Omega^{2,0} & \dots & \xrightarrow{d_H} \Omega^{n-1,0} & \xrightarrow{d_H} \Omega^{n,0} & &
\end{array} \tag{3.24}$$

is the Variational Bicomplex.

Therefore the generalization of (3.15) is:

$$0 \to R \to \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \Omega^{2,0} ... \xrightarrow{d_H} \Omega^{n-1,0} \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{E} F^1 \xrightarrow{\delta_H} F^2 \xrightarrow{\delta_H} F^3 ... \tag{3.25}$$

4. G.L. Problem with non-holonomic constraints

4.1. **G.L. Problem with non-holonomic constraints.** Let us recall from [26] the Generalized Lagrange problem with non-holonomic constraints for $n > 1, m = 1$.

Let us assume $g^\rho(x^i, u, u_{x^j}) = 0$, with $rank[\partial g^\rho/\partial u_{x^j}] = n - l$, $g^\rho(x^i, u, u_{x^j})$ smooth functions with $1 \leq j \leq n$, $1 \leq \rho \leq n-l$ and $l \geq 0$ $(I^*, L^*, \varphi, I^*, L^*)$ is a well-posed valued differential system, where: $I^* =$ span $\{\theta\}$, and $L^* =$ span $\{\theta, dx^i | 1 \leq \ i \leq n\}$

$$\theta = \{du - u_{x^i} dx^i | 1 \leq \ i \leq n\}. \tag{4.1}$$

In this setting we have:

$$d\theta_\mu = -du_{x^\mu} \wedge \omega \tag{4.2}$$
$$d\theta_\rho = -(A^\mu_\rho du_{x^\mu} \wedge \omega - B_\rho \theta \wedge \omega) \tag{4.3}$$
$$A^\mu_\rho = -a^\rho_\sigma g^\sigma_{u_{x^\mu}} \tag{4.4}$$
$$B_\rho = -a^\rho_\sigma g^\sigma_u \tag{4.5}$$

$$L^\mu = (\partial/\partial du_{x^\mu} - A^\mu_\rho \partial/\partial u_{x^\rho})L, \tag{4.6}$$
$$L^{\mu\nu} = (\partial/\partial du_{x^\mu} - A^\mu_\rho \partial/\partial u_{x^\rho})L^\nu, \tag{4.7}$$

and

$$[a^\rho_\sigma] = [g^\sigma_{u_\rho}]^{-1} \text{ with } 1 \leq \rho, \sigma, \leq n-l \text{ and } n-l+1 \leq \mu, \nu \leq n. \tag{4.8}$$

$$\Psi \equiv (L^\mu - \lambda^\mu - \lambda^\rho A^\mu_\rho)\pi^*(du_{x^\mu} \wedge \omega) + d\lambda^\mu \wedge \pi^*(\theta_\mu) + d\lambda^\rho \wedge \pi^*(\theta_\rho)$$
$$+(L_u - \lambda^\rho B_\rho + L_{u_{x^\rho}} B_\rho)\pi^*(\theta \wedge \omega), \tag{4.9}$$

The Cartan system is:

$$\pi^*\theta_\alpha \quad (1 \leq \alpha \leq n) \tag{4.10}$$

$$(L^\mu - \lambda^\mu - \lambda^\rho A^\mu_\rho)\pi^*\omega \quad (n-l+1 \leq \mu \leq n) \tag{4.11}$$

$$(-d\lambda^\rho \wedge \pi^*\omega_\rho - d\lambda^\mu \wedge \pi^*\omega_\mu) + (L_u + \lambda^\rho B_\rho - L_{u_{x^\rho}} B_\rho) \wedge \pi^*\omega \tag{4.12}$$

Let us assume $g^\rho/\partial u_{x^\mu} = 0$ for all $n-l+1 \leq \mu \leq n$ and all $1 \leq \rho \leq n-l$. Then the Euler-Lagrange equation is:

$$E[L] = \partial L/\partial u + \partial L/\partial u_{x^\sigma} B_\sigma - D_{x^\mu}\partial L/\partial u_{x^\mu} + \lambda^\sigma B_\sigma - \lambda^\rho_{x^\rho} \tag{4.13}$$

Proposition 4.1. *Let (I^*, L^*) be a locally embeddable differential system defined in $X = J^1(R^n, R)|_{g^\rho(x^i,u,u_{x^j})=0}$, $rank[\partial g^\rho/\partial u_{x^j}] = n - l$, $g^\rho(x^i, u, u_{x^j})$ smooth functions, $1 \le i, j \le n$, $1 \le \rho \le n - l, l \ge 0$, and $g^\rho/\partial u_{x^\mu} = 0$ for all $n - l + 1 \le \mu \le n$ and all $1 \le \rho \le n - l$, where $I^* =$ span $\{\theta\}$,*
$L^* =$ span $\{\theta, dx^i | 1 \le i \le n\}$,

$$\theta = du - u_{x^j} dx^j \quad 1 \le j \le n. \tag{4.14}$$

Let $Q(x^i, u, u_{x^\mu}, u_{x^\mu x^\nu}, \lambda^\rho, \lambda^\rho_{x^i})$ with $n - l + 1 \le \mu, \nu \le n$, $1 \le \sigma, \rho \le n - l$ and $1 \le i \le n$, with $Q(x^i, u, u_{x^\mu}, t u_{x^\mu x^\nu}, \lambda^\rho, \lambda^\rho_{x^i})$ being a polynomial in t of degree less or equal to 1, and

$$P = Q + \lambda^\rho_{x^\rho} - \lambda^\rho B_\rho. \tag{4.15}$$

And furthermore, let us assume that P satisfies the Helmholtz conditions and does not depend on λ_ρ and $\lambda^\rho_{x^i}$ coordinates, then Q is locally a Euler-Lagrange operator for a Lagrangian $L(x^i, u, u_{x^\mu})$.

Proof: In this case the Helmholtz condition is:

$$\partial P/\partial u_{x^\mu} = D_{x^\mu}\partial P/\partial u_{x^\mu x^\mu} + 1/2 D_{x^\nu}\partial P/\partial u_{x^\mu x^\nu}, \tag{4.16}$$

with $n - l + 1 \le \mu, \nu \le n$.

From Theorem 3.1 we know that a function $F(x^i, u, u_{x^\mu})$ can be found that does not depend on u^ν_{xx}, such that $E[F] = \partial F/\partial u - D^i_x \partial F/\partial u^i_x$.

In addition, if in the domain of P the sequence of spaces is exact:

$$\Omega^{n,0} \xrightarrow{E} F^1 \xrightarrow{H} 0 \tag{4.17}$$

then we have a global solution for the inverse problem.

Example 3. *Let $X = J^1(R^n, R)|_{g^\rho(x^i,u,u_{x^j})=0}$, $rank[\partial g^\rho/\partial u_{x^j}] = n - l$, $g^\rho(x^i, u, u_{x^j})$ smooth functions, $1 \le i, j \le n$, $1 \le \rho \le n - l, l \ge 0$, and $g^\rho/\partial u_{x^\mu} = 0$ for all $n - l + 1 \le \mu \le n$ and all $1 \le \rho \le n - l$, where $I^* =$ span $\{\theta\}$, $L^* =$ span $\{\theta, dx^i | 1 \le i \le n\}$ and $Q(x^i, u, u_{x^\mu}, u_{x^\mu x^\nu}) = 2u_{x^\sigma}(x^i, u)B_\sigma - \sum_\mu 2u_{x^\mu x^\mu} + \lambda^\sigma B_\sigma - \lambda^\rho_{x^\rho}$. Q satisfies Helmhotz equations and is globally a Euler-Lagrange operator with $L = \sum_\mu (u_{x^\mu})^2 + \sum_\sigma (u_{x^\sigma}(x^i, u))^2$.*

References

[1] I. M. Anderson. Natural variational principles on Riemannian manifolds *Ann. of Math.*, Volume 120, pp 329–370, 1984.

[2] I. M. Anderson. Introduction to the variational bicomplex *Contemporary Mathematics*, Volume 132, pp 51–73, 1992.

[3] I. M. Anderson. On the existence of global variational principles *Amer. J. Math.*, Volume 102, pp 781–868, 1980.

[4] I. M. Anderson and G. Thompson. The inverse problem of the calculus of variations for ordinary differential equations *Memoirs Amer. Math. Soc.*,Volume 98, pp vi+110, 1992.

[5] I. M. Anderson. The Variational Bicomplex *http : //bib.tiera.ru/b/* 37499, (book draft, free web version, Utah, 2004), pdf.

[6] R. Bryant, S. S. Chern, R. Gardner and P. Griffiths. Essays on exterior differential systems *Springer-Verlag*, New York, 1990.

[7] C. Caratheodory. Variationsrechnung bei mehrfachen Integralen *Acta Szeged 4*, 1929.

[8] E. Cartan. Les systémes differentielles exterieurs et leurs applicationsgéométriques *Herman*, Paris, 1945.

[9] P. Dedecker. Calcul des variations, formes differentielles et champs geodé siques *Colloques. Internat. du C.N.R.S.*, Strasbourg, 1953.

[10] P. Dedecker. Calcul des variations et topologie algebraique *Mem. Soc. Roy. Sc. Liége*, 4-e série, XIX, fase I, 1957.

[11] P. Dedecker. On the generalization of sympletic geometry to muliple integrals in the calculus of variations *Lecture Notes in Math. Springer*, Vol. 570, Berlin and New York, 1977.

[12] Th. De Donder. Téorie invariantive du calcul de variations *Gautier-Villars*, Paris, 1935.

[13] D. G. B. Edelen. The invariance group for Hamiltonian systems of partial differential equationss *Arch.Rational Mech. Anal.*, Volume 5 pp 95-176, 1961.

[14] D. G. B. Edelen. Nonlocal variations and local invariance of fields *American Elsvier*, New York, 1969.

[15] H. I. Eliasson Variational integrals in fiber bundles *Proc. Sympos. Pure Math. Amer. Math. Soc.*, Volume 16, Providence, RI, pp 67-89 1970.

[16] P. L. Garcia, The Poincaré-Cartan invariant in the calculus of variations *Sympos. Math.*, Volume 14, pp 219-227, 1974.

[17] P. L. Garcia, Gauge algebras, curvature and symplectic structure *J. Differential Geometry*, Volume 12, pp 209-246, 1977.

[18] P. L. Garcia, Critical principal connections and gauge invariance *Rep. Math. Phys*,Volume 13, pp 337-344, 1978.

[19] P. L. Garcia, Tangent structure of Yang-Mills equations and Hodge Theory *Lecture Notes in Math. Springer*, Volume 775, Berlin and New York, 1980.

[20] P. L. Garcia, A. Pérez-Rendón. Symplectic approach to the theory of quantized fields I *Comm. Math Phys.*, Volume 13, pp 24-44, 1969.

[21] P. L. Garcia, A. Pérez-Rendón. Symplectic approch to the theory of quantized fields. II *Arch. Rational Mech. Anal*, Volume 43, pp 101-124, 1971.

[22] P. L. Garcia, A. Pérez-Rendón. Reducibility of the symplectic structure of minimal interactions *Lecture Notes in Math. Springer*, Volume 676, Berlin and New York, pp 409-433, 1978.

[23] H. Goldschmidt, S. Sternberg, The Hamilton-Cartan formalism in the calculus of variations *Ann. Inst. Fourier*, Volume 23, Grenoble, pp 203-267, 1973.

[24] P. Griffiths. Exterior differential systems and the calculus of variations *Birkäuser*, Boston, Basel, Stuttgart, 1983.
[25] C. Günther. The plysymplectic Hamiltonian formalism in the field theory and calculus of variations I: The local case *J. Differential Geometry*, Volume 25, pp 23-53, 1987.
[26] P. G. Henriques. Calculus of variations in the context of exterior differential systems *Differential Geometry and its Applications (North-Holland)*, Volume 3, pp 331–372, 1993.
[27] P. G. Henriques. Well-posed variational problem with mixed endpoint conditions *Differential Geometry and its Applications (North-Holland)*, Volume 3, pp 373–393, 1993.
[28] P. G. Henriques. The Noether theorem and the reduction procedure for the variational calculus in the context of differential systems *C. R. Acad. Sci. Paris Sér. I Math.*, Volume 317, pp 987–992, 1993.
[29] P. G. Henriques. The inverse problem of variational calculus and the problem of mixed endpoint conditions *São Paulo Journal of Mathematical Science*, Volume 2, pp 239–262, 2008.
[30] R. Hermann. Differential geometry and the calculus of varitions *Academic Press*, New York, 1968.
[31] J. Kijowski, W. M. Tulczyjew. A symplectic framework for field theories *Lecture Notes in Math. Springer*, Volume 107, Berlin and New York, 1979.
[32] D. Krupka. Lagrange theory in fibered manifolds *Rep. Math. Phys*, Volume 2, pp 121-133, 1970.
[33] D. Krupka. A geometric theory of ordinary first order variational problems in fibered manifolds I: Critical sections *J. Math Anal. Appl.*, Volume 49, pp 180-206, 1975.
[34] D. Krupka. A geometric theory of ordinary first order variational problems in fibered manifolds II: Invariance *J. Math Anal. Appl.*, Volume 49, pp 469-476, 1975.
[35] T. Lepage. Sur les champs geodésiques des integrals multiples *Bull. Acad. Roy. Belg, CI. Sc.*, 5.éme série 22, 1936.
[36] T. Lepage. Sur les champs geodésiques des integrals multiples *Bull. Acad. Roy. Belg, CI. Sc.*, 5.éme série 27, 1941.
[37] T. Lepage. Champs stationnaires, champs geodésiques et formes integrables I, II *Bull. Acad. Roy. Belg, CI. Sc.*, 5.éme série 28, 1942.
[38] A. Liesen. Feldtheorie in der Variatonrechnung mehfacher Integrale I *Math Annalen*, Volume 171, pp 194-218 1967.
[39] A. Liesen. Feldtheorie in der Variatonrechnung mehfacher Integrale II *Math Annalen*, Volume 171, pp 273-292 1967.
[40] Waldyr M. Oliva and Gláucio Terra. An Inverse Problem on Vakonomic Mechanics *São Paulo Journal of Mathematical Science*, Volume 5, 1, pp 23-35, 2011.
[41] P. J. Olver. Euler operators and conservation laws of BBM equation *Math. Proc. Cam. Phil. Soc.*, Volume 85, pp 143-160, 1979.
[42] P. J. Olver. Applications of Lie groups to differential equations *Springer*, New York, 1986.
[43] R. Ouzilou. Expression symplectic des problems variationnels *Sympos. Math* , Volume 14, pp 85-98, 1972.
[44] H. Rund. The Hamilton-Jacobi theory in the calculus of variations *Van Nostrand*, Princeton, NJ, 1966.
[45] F. Takens. Symmetries, conservation laws and variational principles *Lectures Notes in Mathematics, Springer*, Volume 597, pp 581-603, New York, 1977.

[46] F. Takens. A global version of the inverse problem to the calculus of variations *J. Differential Geometry*, Volume 14, pp 543-562, 1979.

[47] W. M. Tulczyjew. The Euler-Lagrange resolution *Lectures Notes in Mathematics, Springer*, Volume 836, pp 22-48 S, New York, 1980.

[48] W. M. Tulczyjew. Cohomology of the Lagrange complex *Ann. Scuola. Norm. Sup. Pisa*, pp 217-227, 1988.

[49] M. M. Vainberg Variational Methods for tht study of Nonlinear Operators *Holden-Day*, S. Francisco, 1964.

[50] A. M. Vinagradov. The C-spectral sequence, Lagrangian formalism and conservation laws I, II *J. Math Anal. Appl*, Volume 100, pp 1-129, 1984.

[51] A. M. Vinagradov. Symmetries and conservation laws of partial differential equations: basic notions and results *Acta Appl. Math.*, Volume 15, pp 3-22, 1989.

[52] A. M. Vinagradov. Scalar differential invariants, diffiieties and characteristic classes *Mechanics, Analysis and Geometry: 200 Years after Lagrange, M. Francavglia(ed), Elsevier*, Amsterdam, pp 379-416, 1991.

[53] V. Volterra. Leçons sur les Fonctions de Lignes *Gautherier-Villars*, Paris, 1913.

[54] H. Weyl. Geodesic fields in the calculus of variations for multiple integrals *Annals of Math.*, Volume 36, pp 607-629 1935.