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The single-leaf Frobenius Theorem with Applications

Paolo Piccione and Daniel V. Tausk

Dedicated to Prof. Serge Lang

Abstract. Using the notion of Levi form of a smooth distribution, we discuss the local and the global problem of existence of *one* horizontal section of a smooth vector bundle endowed with a horizontal distribution. The analysis will lead to the formulation of a "one-leaf" analogue of the classical Frobenius integrability theorem in elementary differential geometry. Several applications of the result will be discussed. First, we will give a characterization of symmetric connections arising as Levi–Civita connections of semi-Riemannian metric tensors. Second, we will prove a general version of the classical Cartan–Ambrose–Hicks Theorem giving conditions on the existence of an affine map with prescribed differential at one point between manifolds endowed with connections.

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1. Introduction

The central theme of the paper is the study of conditions for the existence of *one* integral leaf of (non integrable) smooth distributions satisfying a given initial condition. The integrability condition given by Frobenius theorem, a very classical result in elementary Differential Geometry, guarantees the existence of integral leaves with *any* initial condition. If on one hand such condition is very strong, on the other hand the involutivity assumption in Frobenius theorem is very restrictive. For instance, the integrability of the horizontal distribution of a connection in a vector bundle is equivalent to the flatness of the connection.

A measure of non integrability for a smooth distribution $\mathcal{D}$ on a manifold E is provided by the so-called *Levi form* $\mathfrak{L}^{\mathcal{D}}$ of $\mathcal{D}$; this is a skew-symmetric bilinear tensor defined on the distribution, taking values in the quotient $TE/\mathcal{D}$. For $x \in E$ and $v, w \in \mathcal{D}_x$, the value $\mathfrak{L}^{\mathcal{D}}_x(v,w)$ is given by the projection on $T_xE/\mathcal{D}_x$ of the Lie bracket $[X,Y]_x$, where X and Y are arbitrary extensions of v and w respectively to $\mathcal{D}$-horizontal vector fields. If $\Sigma \subset E$ is an integral submanifold of $\mathcal{D}$, then the Levi form of $\mathcal{D}$ vanishes on the points of Σ. The first central observation that is made in this paper is that, conversely, given an immersed submanifold Σ of E with $T_{x_0}\Sigma = \mathcal{D}_{x_0}$ for some $x_0 \in E$, if Σ is *ruled* (in an appropriate sense) by curves tangent to $\mathcal{D}$, and if $\mathfrak{L}^{\mathcal{D}}$ vanishes along Σ, then Σ is an integral submanifold of $\mathcal{D}$. In particular, assume that $\mathcal{D} \subset TE$ is a horizontal distribution of a vector bundle $\pi : E \to M$ over a manifold M, and that Σ is a local section of π which is obtained by parallel lifting of a family of curves on M issuing from some fixed point x_0. If the Levi form of $\mathcal{D}$ vanishes along Σ, then Σ is a parallel section of π (Theorem 2.5); we call this result the *(local) single leaf Frobenius theorem.* In the real analytic case, a higher order version of this result is given in Theorem 2.7; roughly speaking, the higher order derivatives of the Levi form $\mathfrak{L}^{\mathcal{D}}$ are obtained from iterated Lie brackets of $\mathcal{D}$-horizontal vector fields. The higher order single-leaf Frobenius theorem states that, in the real-analytic case, if at some point x_0 of the manifold E all the iterated brackets of vector fields in $\mathcal{D}$ belong to $\mathcal{D}_{x_0}$, then there exists an integral submanifold of $\mathcal{D}$ through x_0 (see [7]).

A global version of the single-leaf Frobenius theorem is discussed in Theorem 3.11; here, the base manifold M has to be assumed simply-connected. Assume that a spray is given on M, for instance, the geodesic spray of some Riemannian metric. The existence of a global parallel section of π through a point e_0 with $\pi(e_0) = x_0 \in M$ is guaranteed by the following condition: every piecewise solution $\gamma : [a, b] \to M$ of $\mathcal{S}$ with $\gamma(a) = x_0$ should admit a parallel lifting $\widetilde{\gamma} : [a, b] \to E$ such that $\widetilde{\gamma}(a) = e_0$ and such that the Levi form of $\mathcal{D}$ vanishes at the point $\widetilde{\gamma}(b)$.

We also observe (Proposition 3.12) that in the real analytic case, every local parallel section defined on a non empty open subset of a simply connected manifold M extends to a global parallel section.

Reference [4] is an excellent reading for those who are interested in more general versions of the single–leaf Frobenius theorem, which is discussed in the case that:

- the distribution $\mathcal{D}$ is not assumed to have constant rank;
- the manifold M is allowed to be infinite dimensional (Banach manifold).

A huge number of problems in Analysis and in Geometry can be cast into the language of distributions and integral submanifolds. As an application of the theory discussed in this paper, we will consider two problems. First, we will characterize those symmetric connections that are Levi–Civita connections of some semi-Riemannian metric (alternatively, this problem can be studied using holonomy theory, see [2]). Second, we will prove a very general version of another classical result in Differential Geometry, which is the Cartan–Ambrose–Hicks theorem (see [1, 5, 8]). We will prove a necessary and sufficient condition for the existence of an affine map between manifolds endowed with arbitrary connections.

Let us describe briefly these two results.

Consider the case of a distribution given by the horizontal space of a connection ∇ of a vector bundle $\pi : E \to M$. For $\xi \in E$, set $m = \pi(\xi) \in M$ and $E_m = \pi^{-1}(m)$; one can identify $\mathcal{D}_\xi$ with T_mM, and the quotient $T_\xi E/\mathcal{D}_\xi$ with the vertical subspace $T_\xi(E_m) \cong E_m$. Then, the Levi form $\mathfrak{L}^{\mathcal{D}}_\xi : T_mM \times T_mM \to E_m$ is given by the curvature tensor of ∇, up to a sign (Lemma 4.1). In this case, the single leaf Frobenius theorem tells us that a local parallel section of π through some point $e_0 \in E$ exists provided that along each parallel lifting of a family of curves issuing from $\pi(e_0) \in M$ the curvature tensor vanishes (Corollary 4.2). In the real analytic case, the existence of a local parallel section through a point $\xi \in E$ is equivalent to the vanishing of all the covariant derivatives $\nabla^k R$, $k \geq 0$, of the curvature tensor R at the point $m = \pi(\xi)$ (Proposition 4.5).

Assume that the vector bundle $\pi : E \to M$ is endowed with a connection ∇, and denote by ∇^{bil} the induced connection on $E^* \otimes E^*$. If g is a (local) section of $E^* \otimes E^*$, then vanishing of the curvature tensor R^{bil} of ∇^{bil} means that the bilinear map $g\big(R(v,w)\cdot,\cdot\big)$ is anti-symmetric for all v, w (formula (13)). From this observation, we get the following result on the existence of parallel metric tensor relatively to a given connection ∇ on a manifold M: given a nondegenerate (symmetric) bilinear form g_0 on $T_{m_0}M$, assume that the tensor g obtained from g_0 by ∇-parallel transport along a family of curves issuing from m_0 is such that R is g-anti-symmetric. Then, g is ∇-parallel (Proposition 4.6). Similarly, in the real analytic case, if $\nabla^k R$ at m_0 is g_0-anti-symmetric for all $k \geq 0$, then g_0 extends to a semi-Riemannian metric tensor whose Levi–Civita connection is ∇. These results have been used in [6] to obtain characterizations of left-invariant semi-Riemannian Levi Civita connections in Lie groups.

As another application of our theory, in Section 5 we will study the problem of existence of an affine (i.e., connection preserving) map f between two affine manifolds (M, ∇^M) and (N, ∇^N), whose value $y_0 \in N$ at some point $x_0 \in M$ is given and whose differential $\mathrm{d}f(x_0) : T_{x_0}M \to T_{y_0}N$ is prescribed. We prove a general version of the classical Cartan–Ambrose–Hicks theorem (Theorem 5.1 for the local result, Theorem 5.3 for the global version), giving a necessary and sufficient condition for the existence of such a map; here, the connections ∇^M and ∇^N are not assumed to be symmetric, and no assumption is made on the dimension of the manifolds M and N, as well as on the linear map $\mathrm{d}f(x_0)$. The key observation here (Lemma 5.6) is that, considering the vector bundle $E = \mathrm{Lin}(TM, TN)$ over the product $M \times N$, endowed with a natural connection induced by ∇^M and ∇^N (see formula (15)), then a smooth map $f : U \subset M \to N$ is an affine map if and only if the differential $\mathrm{d}f$ is a local parallel section of E along the map $U \ni x \mapsto \big(x, f(x)\big) \in M \times N$. When M and N are endowed with semi-Riemannian metrics and ∇^M and ∇^N are the respective Levi–Civita connections, then our result gives a necessary and sufficient condition for the existence of a totally geodesic immersion of M in N.

The proof of the Cartan–Ambrose–Hicks theorem is obtained as an application of the single-leaf Frobenius theorem, once the Levi form of the horizontal distribution of the induced connection on E is computed (Lemma 5.14). The higher order version of this result (Theorem 5.16) is particularly interesting: in the real analytic case, a (local) affine map $f : U \subset M \to N$ with $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma$ exists if and only if σ relates covariant derivatives of all order of curvature and torsion of ∇^M and ∇^N at the points x_0 and y_0 respectively.

As a nice corollary of the higher order Cartan–Ambrose–Hicks theorem, we get the following curious result (Corollary 5.19): if M is a real-analytic manifold endowed with a real-analytic connection ∇, and let $x_0 \in M$ be fixed; there exists an affine symmetry around x_0 if and only if $\nabla^{(2r)}T_{x_0} = 0$ and $\nabla^{(2r+1)}R_{x_0} = 0$ for all $r \geq 0$.

A certain effort has been made in order to make the presentation of the material self-contained. For this reason, we have found convenient to discuss, together with the original material, some auxiliary topics needed for a more complete presentation of our results. For instance, in Subsection 3.1, we discuss and give the basic properties of the exponential map of a spray (this is needed in our statement of the global one-leaf Frobenius theorem). Similarly, in Appendix B we develop the basic theory needed for making computations with covariant derivatives, curvatures and torsions of connections on vector bundles obtained by functorial constructions; this kind of computations is heavily used throughout the paper. Finally, in Appendix A we discuss a globalization principle in a very general setting of pre-sheafs on topological spaces. Such principle is used in the proof of the global versions of the single-leaf Frobenius theorem (see for instance the proofs of Theorem 3.11 and Proposition 3.12). Typically, the globalization principle is employed in the following manner: given a vector bundle $\pi : E \to M$, a pre-sheaf $\mathfrak{P}$ is defined on M by defining, for all open subset $U \subset M$, $\mathfrak{P}(U)$ to be the set of all sections $s : U \to E$ of π satisfying some property (for instance, parallel sections). For $V \subset U$, and $s \in \mathfrak{P}(U)$, the map $\mathfrak{P}_{U,V} : \mathfrak{P}(U) \to \mathfrak{P}(V)$ is given by setting $\mathfrak{P}_{U,V}(s) = s|_V$. In this context, the existence of a global section of π with the required property is equivalent to the fact that the set $\mathfrak{P}(M)$ should be non empty. The central result of Appendix A (Proposition A.8) gives a sufficient condition for this, in terms of three properties of pre-sheaves, called *localization*, *uniqueness* and *extension*.

Dedicatory. The proof of the single-leaf Frobenius theorem discussed here has taken inspiration from the proof of the classical Frobenius theorem presented in Serge Lang's world famous book [3]. Since the very beginning of their mathematical careers, both authors have benefited very much from this and from other beautiful books published by Prof. Lang. We want to thank him by dedicating this paper to his memory.

2. The Levi form and the "single leaf Frobenius Theorem"

Recall that a *smooth distribution* $\mathcal{D}$ on a smooth manifold E is a smooth vector subbundle of the tangent bundle TE. For $x \in E$ we set $\mathcal{D}_x =$

$T_xE \cap \mathcal{D}$, i.e., $\mathcal{D}_x$ is the fiber of the vector bundle $\mathcal{D}$ over x. A vector field X on E is called *horizontal* with respect to a distribution $\mathcal{D}$ (or simply *$\mathcal{D}$-horizontal*) if X takes values in $\mathcal{D}$, i.e., if $X(x) \in \mathcal{D}_x$ for all $x \in E$. An immersed submanifold S of E is called an *integral submanifold* for $\mathcal{D}$ if $T_xS = \mathcal{D}_x$, for all $x \in S$. The distribution $\mathcal{D}$ is called *integrable* if through every point of E passes an integral submanifold for $\mathcal{D}$.

2.1. The Levi form of a smooth distribution.

Definition 2.1. Let E be a smooth manifold and let $\mathcal{D}$ be a distribution on E. The *Levi form* of $\mathcal{D}$ at a point $x \in E$ is the bilinear map:

$$\mathfrak{L}_x^{\mathcal{D}} : \mathcal{D}_x \times \mathcal{D}_x \longrightarrow T_xE/\mathcal{D}_x$$

defined by $\mathfrak{L}_x^{\mathcal{D}}(v,w) = [X,Y](x) + \mathcal{D}_x \in T_xE/\mathcal{D}_x$, where X and Y are $\mathcal{D}$-horizontal smooth vector fields defined in an open neighborhood of x in E with $X(x) = v$ and $Y(x) = w$. By $[X,Y]$ we denote the Lie bracket of the vector fields X and Y.

Below we show that the Levi form is well-defined, i.e., $[X,Y](x) + \mathcal{D}_x$ does not depend on the choice of the $\mathcal{D}$-horizontal vector fields X and Y with $X(x) = v$, $Y(x) = w$. Let θ be a smooth $\mathbb{R}^k$-valued 1-form on an open neighborhood U of x such that $\mathrm{Ker}(\theta_x) = \mathcal{D}_x$ for all $x \in U$. If X and Y are vector fields on an open neighborhood of x then Cartan's formula for exterior differentiation gives:

$$\mathrm{d}\theta(X,Y) = X\big(\theta(Y)\big) - Y\big(\theta(X)\big) - \theta\big([X,Y]\big).$$

If X and Y are $\mathcal{D}$-horizontal then the equality above reduces to:

$$\mathrm{d}\theta(X,Y) = -\theta\big([X,Y]\big).$$

The formula above implies that if X', Y' are $\mathcal{D}$-horizontal vector fields such that $X'(x) = X(x)$ and $Y'(x) = Y(x)$ then $\theta\big([X,Y] - [X',Y']\big)(x) = 0$, i.e., $[X,Y](x) - [X',Y'](x) \in \mathcal{D}_x$. Hence the Levi form is well-defined. Setting $X(x) = v$ and $Y(x) = w$ we obtain the following formula:

$$\bar{\theta}_x\big(\mathfrak{L}_x^{\mathcal{D}}(v,w)\big) = -\mathrm{d}\theta(v,w), \quad v,w \in \mathcal{D}_x, \tag{1}$$

where $\bar{\theta}_x : T_xE/\mathcal{D}_x \to \mathbb{R}^k$ denotes the linear map induced by θ_x in the quotient space.

Remark 2.2. Clearly, by the classical Frobenius Theorem, $\mathcal{D}$ is integrable if and only if its Levi form is identically zero. Moreover, the Levi form of $\mathcal{D}$ vanishes along any integral submanifold of $\mathcal{D}$.

2.1. **Example.** Let U be an open subset of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ and let

$$F : U \ni (x,y) \longmapsto F_{(x,y)} \in \mathrm{Lin}(\mathbb{R}^k, \mathbb{R}^{n-k})$$

be a smooth map. We consider the distribution $\mathcal{D} = \mathrm{Gr}(F)$ on U, i.e., $\mathcal{D}_{(x,y)} = \mathrm{Gr}\big(F_{(x,y)}\big)$, for all $(x,y) \in U$. Given $X \in \mathbb{R}^k$, we define a $\mathcal{D}$-horizontal vector field $\widetilde{X}$ on U by setting $\widetilde{X}_{(x,y)} = \big(X, F_{(x,y)}(X)\big)$, for all $(x,y) \in U$. Given $X, Y \in \mathbb{R}^k$ then:

$$[\widetilde{X}, \widetilde{Y}] = \big(0, \partial_x F(X,Y) + \partial_y F(F(X),Y) - \partial_x F(Y,X) + \partial_y F(F(Y),X)\big).$$

If we identify $\mathcal{D}_{(x,y)}$ with $\mathbb{R}^k$ by the isomorphism $\big(X, F(X)\big) \mapsto X$ and $\mathbb{R}^n/\mathcal{D}_{(x,y)}$ with $\mathbb{R}^{n-k} \cong \{0\}^k \times \mathbb{R}^{n-k}$ by the isomorphism $(v,w) + \mathcal{D}_{(x,y)} \mapsto w - F(v)$ then the Levi form $\mathfrak{L}^{\mathcal{D}} : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^{n-k}$ is given by:

$$\mathfrak{L}^{\mathcal{D}}(X,Y) = [\widetilde{X}, \widetilde{Y}]. \tag{2}$$

Lemma 2.3. *Let E be a smooth manifold, $\mathcal{D}$ be a smooth distribution on E and let*

$$U \ni (t,s) \longmapsto H(t,s) \in E$$

be a smooth map defined on an open subset $U \subset \mathbb{R}^2$. Let $I \subset \mathbb{R}$ be an interval and let $s_0 \in \mathbb{R}$ be such that $I \times \{s_0\} \subset U$ and $\mathfrak{L}^{\mathcal{D}}_{H(t,s_0)} = 0$ for all $t \in I$. Assume that $\frac{\partial H}{\partial t}(t,s) \in \mathcal{D}$ for all $(t,s) \in U$. If $\frac{\partial H}{\partial s}(t_0, s_0) \in \mathcal{D}$ for some $t_0 \in I$ then $\frac{\partial H}{\partial s}(t, s_0) \in \mathcal{D}$ for all $t \in I$.

Proof. The set:

$$I' = \big\{ t \in I : \tfrac{\partial H}{\partial s}(t, s_0) \in \mathcal{D} \big\}$$

is obviously closed in I because the map $I \ni t \mapsto \frac{\partial H}{\partial s}(t, s_0) \in TE$ is continuous and $\mathcal{D}$ is a closed subset of TE. Since I is connected and $t_0 \in I'$, the proof will be complete once we show that I' is open in I. Let $t_1 \in I'$ be fixed. Let θ be an $\mathbb{R}^k$-valued smooth 1-form defined in an open neighborhood V of $H(t_1, s_0)$ in E such that the linear map $\theta_x : T_x E \to \mathbb{R}^k$ is surjective and $\mathrm{Ker}(\theta_x) = \mathcal{D}_x$ for all $x \in V$. Choose a distribution $\mathcal{D}'$ on V such that $T_x E = \mathcal{D}_x \oplus \mathcal{D}'_x$ for all $x \in V$. Then, for each $x \in V$, θ_x restricts to an isomorphism from $\mathcal{D}'_x$ onto $\mathbb{R}^k$. Let J be a connected neighborhood of t_1 in I such that $H(t, s_0) \in V$ for all $t \in J$. We will show below that the map:

$$J \ni t \longmapsto \theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial s}(t, s_0)\big) \in \mathbb{R}^k \tag{3}$$

is a solution of a homogeneous linear ODE; since $\theta_{H(t_1,s_0)}\big(\frac{\partial H}{\partial s}(t_1, s_0)\big) = 0$, it will follow that $\theta_{H(t,s_0)}\big(\frac{\partial H}{\partial s}(t, s_0)\big) = 0$ for all $t \in J$, i.e., $J \subset I'$.

We denote by $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ the canonical basis of $\mathbb{R}^2$ and we apply Cartan's formula for exterior differentiation to the 1-form $H^*\theta$ obtaining:

$$\mathrm{d}(H^*\theta)\big(\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial s}\big) = \tfrac{\partial}{\partial t}\Big((H^*\theta)\big(\tfrac{\partial}{\partial s}\big)\Big) - \tfrac{\partial}{\partial s}\Big((H^*\theta)\big(\tfrac{\partial}{\partial t}\big)\Big) - (H^*\theta)\big(\big[\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial s}\big]\big).$$

Since $\mathrm{d}(H^*\theta) = H^*(\mathrm{d}\theta)$ and $\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = 0$ we get:

$$(4)\quad \mathrm{d}\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial t}(t,s_0), \tfrac{\partial H}{\partial s}(t,s_0)\big) = \tfrac{\partial}{\partial t}\Big(\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial s}(t,s_0)\big)\Big) - \tfrac{\partial}{\partial s}\Big|_{s=s_0}\Big(\theta_{H(t,s)}\big(\tfrac{\partial H}{\partial t}(t,s)\big)\Big), \quad t \in J.$$

Observe that, since $\frac{\partial H}{\partial t}(t,s)$ is in $\mathcal{D}$, the last term on the righthand side of (4) vanishes. We can write $\frac{\partial H}{\partial s}(t,s_0) = u_1(t) + u_2(t)$ with $u_1(t) \in \mathcal{D}$ and $u_2(t) \in \mathcal{D}'$. Since the Levi form of $\mathcal{D}$ vanishes at points of the form $H(t,s_0)$, equation (1) implies that $\mathrm{d}\theta_{H(t,s_0)}(v,w) = 0$ for all $v, w \in \mathcal{D}_{H(t,s_0)}$. We may thus replace $\frac{\partial H}{\partial s}(t,s_0)$ by $u_2(t)$ in the lefthand side of (4). For $t \in J$ we consider the linear map $L(t) : \mathbb{R}^k \to \mathbb{R}^k$ defined by:

$$L(t)\cdot z = \mathrm{d}\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial t}(t,s_0), \sigma_{H(t,s_0)}(z)\big), \quad z \in \mathbb{R}^k,$$

where, for $x \in V$, $\sigma_x : \mathbb{R}^k \to \mathcal{D}'_x$ denotes the inverse of the isomorphism

$$\theta_x|_{\mathcal{D}'_x} : \mathcal{D}'_x \longrightarrow \mathbb{R}^k.$$

Observe that:

$$\begin{aligned}\mathrm{d}\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial t}(t,s_0), \tfrac{\partial H}{\partial s}(t,s_0)\big) &= \mathrm{d}\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial t}(t,s_0), u_2(t)\big)\\ &= L(t)\cdot\theta_{H(t,s_0)}\big(u_2(t)\big)\\ &= L(t)\cdot\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial s}(t,s_0)\big).\end{aligned}$$

Equation (4) can now be rewritten as:

$$\tfrac{\partial}{\partial t}\Big(\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial s}(t,s_0)\big)\Big) = L(t)\cdot\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial s}(t,s_0)\big), \quad t \in J.$$

Hence the map (3) is a solution of a homogeneous linear ODE and we are done. □

2.2. **Horizontal distributions and horizontal liftings.** If E, M are smooth manifolds and $\pi : E \to M$ is a smooth submersion then a smooth distribution $\mathcal{D}$ on E is called *horizontal* with respect to π if

$$T_xE = \mathrm{Ker}(\mathrm{d}\pi_x) \oplus \mathcal{D}_x,$$

for all $x \in E$. Given a smooth horizontal distribution $\mathcal{D}$ on E then a piecewise smooth curve $\tilde\gamma : I \to E$ is called *horizontal* if $\tilde\gamma'(t) \in \mathcal{D}$ for all t for which $\tilde\gamma'(t)$ exists. Given a piecewise smooth curve $\gamma : I \to M$ then a *horizontal lifting* of γ is a horizontal piecewise smooth curve $\tilde\gamma : I \to E$ such that $\pi\circ\tilde\gamma = \gamma$.

By standard results of existence and uniqueness of solutions of ODE's it follows that given $t_0 \in I$ and $x_0 \in \pi^{-1}\big(\gamma(t_0)\big)$ then there exists a unique maximal horizontal lifting $\tilde\gamma$ of γ with $\tilde\gamma(t_0) = x_0$ defined in a subinterval of I around t_0.

Let Λ be a smooth manifold. By a *Λ-parametric family of curves ψ* on M we mean a smooth map $\psi : Z \subset \mathbb{R} \times \Lambda \to M$ defined on an open subset Z of $\mathbb{R} \times \Lambda$ such that the set:

$$I_\lambda = \{t \in \mathbb{R} : (t, \lambda) \in Z\} \subset \mathbb{R}$$

is an interval containing the origin, for all $\lambda \in \Lambda$. By a *local right inverse* of ψ we mean a locally defined smooth map $\alpha : V \subset M \to Z$ such that $\psi(\alpha(m)) = m$, for all $m \in V$.

2.2. **Example.** Let M be a smooth manifold endowed with a connection ∇. Given a point $x_0 \in M$ we set $\Lambda = T_{x_0}M$ and we define a Λ-parametric family of curves ψ on M by setting $\psi(t, \lambda) = \exp_{x_0}(t\lambda)$; the domain $Z \subset \mathbb{R} \times \Lambda$ of ψ is the set of pairs (t, λ) such that $t\lambda$ is in the domain of $\exp_{x_0}$. A local right inverse of ψ is defined as follows: let V_0 be an open neighborhood of the origin in $T_{x_0}M$ that is mapped diffeomorphically by $\exp_{x_0}$ onto an open neighborhood V of x_0 in M. We set:

$$\alpha(m) = \big(1, (\exp_{x_0}|_{V_0})^{-1}(m)\big),$$

for all $m \in V$. We remark that the same construction holds if one replaces the geodesic spray of a connection with an arbitrary spray (see Section 3).

A *local section* of a smooth submersion $\pi : E \to M$ is a locally defined smooth map $s : U \subset M \to E$ such that $\pi \circ s = \mathrm{Id}_U$. A local section s is called *horizontal* if the range of $\mathrm{d}s(m)$ is $\mathcal{D}_{s(m)}$, for all $m \in U$.

Lemma 2.4. *Let $s_1 : U \to E$, $s_2 : U \to E$ be local smooth horizontal sections of E defined in an open connected subset U of M. If $s_1(x) = s_2(x)$ for some $x \in U$ then $s_1 = s_2$.*

Proof. Given $y \in U$, there exists a piecewise smooth curve $\gamma : [a, b] \to U$ with $\gamma(a) = x$ and $\gamma(b) = y$. Then $s_1 \circ \gamma$ and $s_2 \circ \gamma$ are both horizontal liftings of γ starting at the same point of E; hence $s_1 \circ \gamma = s_2 \circ \gamma$ and $s_1(y) = s_2(y)$. □

2.3. **Example.** Consider the distribution $\mathcal{D} = \mathrm{Gr}(F)$ on $U \subset \mathbb{R}^n$ defined in Example 2.1. Then the first projection $\pi_1 : U \to \mathbb{R}^k$ is a submersion and $\mathcal{D}$ is horizontal with respect to π_1. A horizontal section $s : \mathbb{R}^k \supset V \to \mathbb{R}^n$ of π_1 is a map $s(x) = (x, f(x))$ where $f : V \to \mathbb{R}^{n-k}$ is a solution of the *total differential equation*:

$$\mathrm{d}f(x) = F\big(x, f(x)\big), \quad x \in V. \tag{5}$$

2.3. **The single leaf Frobenius theorem.**

Theorem 2.5 (local single leaf Frobenius). *Let E, M be smooth manifolds, $\pi : E \to M$ be a smooth submersion, $\mathcal{D}$ be a smooth horizontal distribution* *on E and $\psi : Z \subset \mathbb{R} \times \Lambda \to M$ be a Λ-parametric family of curves on M with a local right inverse $\alpha : V \subset M \to Z$. Let $\tilde{\psi} : Z \to E$ be a Λ-parametric family of curves on E such that $t \mapsto \tilde{\psi}(t, \lambda)$ is a horizontal lifting of $t \mapsto \psi(t, \lambda)$, for all $\lambda \in \Lambda$. Assume that:*

(a) *the Levi form of $\mathcal{D}$ vanishes on the range of $\tilde{\psi}$;*

(b) *$\partial_\lambda \tilde{\psi}(0, \lambda) : T_\lambda \Lambda \to T_{\tilde{\psi}(0,\lambda)} E$ takes values in $\mathcal{D}$ for all $\lambda \in \Lambda$.*

Then $s = \tilde{\psi} \circ \alpha : V \to E$ is a local horizontal section of π.

Proof. If $]-\varepsilon, \varepsilon[\ni s \mapsto \lambda(s)$ is an arbitrary smooth curve on Λ then the map

$$H(t, s) = \tilde{\psi}\big(t, \lambda(s)\big)$$

satisfies the hypotheses of Lemma 2.3 with $t_0 = 0$ and $s_0 = 0$. Thus:

$$\frac{\partial H}{\partial s}(t, 0) = \frac{\partial \tilde{\psi}}{\partial \lambda}\big(t, \lambda(0)\big)\lambda'(0)$$

is in $\mathcal{D}$ for all $t \in I_{\lambda(0)}$. It follows that $\mathrm{d}\tilde{\psi}_{(t,\lambda)}$ takes values in $\mathcal{D}$, for all $(t, \lambda) \in Z$. Hence $\mathrm{d}s(m) = \mathrm{d}\tilde{\psi}\big(s(m)\big) \circ \mathrm{d}\alpha(m)$ also takes values in $\mathcal{D}$, for all $m \in V$. □

Remark 2.6. We observe that if the map $\lambda \mapsto \tilde{\psi}(0, \lambda)$ is constant then hypothesis (b) of Theorem 2.5 is automatically satisfied. Theorem 2.5 is typically used as follows: one considers the Λ-parametric family of curves ψ explained in Example 2.2, a fixed point $e_0 \in \pi^{-1}(x_0) \subset E$ and for each $\lambda \in \Lambda$ one defines $t \mapsto \tilde{\psi}(t, \lambda)$ to be the horizontal lifting of $t \mapsto \psi(t, \lambda)$ with $\tilde{\psi}(0, \lambda) = e_0$.

2.4. **Example.** The single leaf Frobenius theorem can be used to prove the existence of solutions of the total differential equation (5) satisfying a initial condition $f(x_0) = y_0$ as follows. Let V be a star-shaped open neighborhood of x_0 in $\mathbb{R}^k$. Set $\Lambda = \mathbb{R}^k$; we define a Λ-parametric family of curves $\psi : Z \subset \mathbb{R} \times \Lambda \to M$ on $M = \mathbb{R}^k$ by setting $\psi(t, \lambda) = x_0 + t\lambda$, where Z is the set of pairs (t, λ) with $x_0 + t\lambda \in V$. A horizontal lifting $t \mapsto \tilde{\psi}(t, \lambda) = \big(\psi(t, \lambda), \Psi(t, \lambda)\big)$ of the curve $t \mapsto \psi(t, \lambda)$ is a solution of the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t, \lambda) = F_{\tilde{\psi}(t,\lambda)}(\lambda). \tag{6}$$

We choose the solution $t \mapsto \tilde{\psi}(t, \lambda)$ of the ODE (6) with initial condition $\Psi(0, \lambda) = y_0$. We can assume that V is small enough so that $\tilde{\psi}$ is well-defined on Z. Hypothesis (b) of Theorem 2.5 is then automatically satisfied and hypothesis (a) is equivalent to the condition that (2) vanishes on the points of the form $\tilde{\psi}(t, \lambda)$, $(t, \lambda) \in Z$. Under these circumstances, the thesis of Theorem 2.5 guarantees that $f : V \ni x \mapsto \Psi(1, x - x_0) \in \mathbb{R}^{n-k}$ is a solution of the total differential equation (5) with $f(x_0) = y_0$.

2.4. **The higher order single leaf Frobenius theorem.** Let $\mathcal{D}$ be a smooth distribution on a smooth manifold E. We denote by $\Gamma(TE)$ the set of all smooth vector fields on E, by $\Gamma(\mathcal{D})$ the subspace of $\Gamma(TE)$ consisting of $\mathcal{D}$-horizontal vector fields and by $\Gamma^\infty(\mathcal{D})$ the Lie subalgebra of $\Gamma(TE)$ spanned by $\Gamma(\mathcal{D})$. The Lie algebra $\Gamma^\infty(\mathcal{D})$ can be alternatively described as follows; we define recursively a sequence

$$\Gamma^0(\mathcal{D}) \subset \Gamma^1(\mathcal{D}) \subset \Gamma^2(\mathcal{D}) \subset \cdots$$

of subspaces of $\Gamma(TE)$ by setting $\Gamma^0(\mathcal{D}) = \Gamma(\mathcal{D})$ and $\Gamma^{r+1}(\mathcal{D})$ to be the subspace of $\Gamma(TE)$ spanned by $\Gamma^r(\mathcal{D})$ and by the brackets $[X, Y]$, with $X \in \Gamma^r(\mathcal{D})$ and $Y \in \Gamma(\mathcal{D})$. Then:

$$\Gamma^\infty(\mathcal{D}) = \bigcup_{r=0}^{\infty} \Gamma^r(\mathcal{D}).$$

Given $X \in \Gamma(TE)$ we denote by $\mathrm{ad}_X : \Gamma(TE) \to \Gamma(TE)$ the operator $\mathrm{ad}_X(Y) = [X, Y]$.

Theorem 2.7. *Let E be a real-analytic manifold endowed with a real-analytic distribution $\mathcal{D}$. Given $e_0 \in E$ then there exists an integral submanifold of $\mathcal{D}$ passing through e_0 if and only if $X(e_0) \in \mathcal{D}_{e_0}$, for all $X \in \Gamma^\infty(\mathcal{D})$.*

Proof. If there exists an integral submanifold S of $\mathcal{D}$ passing through e_0 then it follows immediately by induction on r that $X(S) \subset \mathcal{D}$, for all $X \in \Gamma^r(\mathcal{D})$ and all $r \geq 0$. Thus, $X(e_0) \in \mathcal{D}_{e_0}$, for all $X \in \Gamma^\infty(\mathcal{D})$. Conversely, assume that $X(e_0) \in \mathcal{D}_{e_0}$, for all $X \in \Gamma^\infty(\mathcal{D})$. By considering a convenient real-analytic local chart around e_0 we may assume without loss of generality that $E = U$ is an open subset of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ and that $\mathcal{D}$ is of the form $\mathrm{Gr}(F)$ (see Example 2.1). Write $e_0 = (x_0, y_0)$; we will use the ideas explained in Example 2.4 to find a solution f of the total differential equation (5) with $f(x_0) = y_0$. Then $\mathrm{Gr}(f)$ is the required integral submanifold of $\mathcal{D}$ passing through e_0. Observe that given $\lambda \in \mathbb{R}^k$ then $t \mapsto \psi(t, \lambda)$ is an integral curve of the constant vector field λ in $\mathbb{R}^k$ and thus the horizontal lift $t \mapsto \tilde{\psi}(t, \lambda)$ is an integral curve of the vector field $\tilde{\lambda} = \big(\lambda, F(\lambda)\big)$ on $\mathbb{R}^n$ passing through e_0 at $t = 0$. We now let $\lambda \in \mathbb{R}^k$, $X, Y \in \mathbb{R}^k$, be fixed and we define a map $t \mapsto \phi(t) \in \mathbb{R}^{n-k}$ by setting:

$$\phi(t) = \mathfrak{L}^{\mathcal{D}}_{\tilde{\psi}(t,\lambda)}(X, Y) = [\widetilde{X}, \widetilde{Y}]_{\psi(t,\lambda)}.$$

The proof will be completed once we show that ϕ is identically zero; since ϕ is real-analytic, it suffices to proof that all derivatives of ϕ at $t = 0$ vanish.

Let us show by induction on r that for all $r \geq 0$ the r-th derivative of ϕ is given by:

$$(7) \qquad \phi^{(r)}(t) = (\mathrm{ad}_{\tilde{\lambda}})^r[\widetilde{X}, \widetilde{Y}] + L^{(r)}\big((\mathrm{ad}_{\tilde{\lambda}})^i[\widetilde{X}, \widetilde{Y}];\ i = 0, 1, \ldots, r-1\big),$$

where the righthand side is computed at the point $\tilde{\psi}(t, \lambda)$ and $L^{(r)}$ is a smooth map that associates to each $(x, y) \in U \subset \mathbb{R}^n$ a linear map:

$$L^{(r)}_{(x,y)} : \bigoplus_r \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^{n-k}.$$

From equality (7) the conclusion will follow; namely, for all i, $(\mathrm{ad}_{\tilde{\lambda}})^i[\widetilde{X}, \widetilde{Y}]$ is in $\{0\}^k \times \mathbb{R}^{n-k}$ and since $\big((\mathrm{ad}_{\tilde{\lambda}})^i[\widetilde{X}, \widetilde{Y}]\big)_{e_0} \in \mathcal{D}_{e_0}$, we get $\big((\mathrm{ad}_{\tilde{\lambda}})^i[\widetilde{X}, \widetilde{Y}]\big)_{e_0} = 0$. Hence $\phi^{(r)}(0) = 0$, for all $r \geq 0$. To prove (7) simply differentiate both sides with respect to t, observing that:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{ad}_{\tilde{\lambda}})^i[\widetilde{X}, \widetilde{Y}] = \mathrm{d}\big((\mathrm{ad}_{\tilde{\lambda}})^i[\widetilde{X}, \widetilde{Y}]\big) \cdot \tilde{\lambda} = (\mathrm{ad}_{\tilde{\lambda}})^{i+1}[\widetilde{X}, \widetilde{Y}] + \mathrm{d}\tilde{\lambda}\big((\mathrm{ad}_{\tilde{\lambda}})^i[\widetilde{X}, \widetilde{Y}]\big).$$

□

Remark 2.8. Clearly, the hypotheses of Theorem 2.7 are local, i.e., if U is an open neighborhood of e_0 in E then $X(e_0) \in \mathcal{D}_{e_0}$ for all $X \in \Gamma^\infty(\mathcal{D}|_U)$ if and only if $X(e_0) \in \mathcal{D}_{e_0}$ for all $X \in \Gamma^\infty(\mathcal{D})$. Replacing E with an open neighborhood of e_0, we may assume that $\mathcal{D}$ admits a global referential X_1, $\ldots$, X_k. It is easy to see that $\Gamma^r(\mathcal{D})$ is the $C^\infty(E)$-module spanned by X_1, $\ldots$, X_k and by the iterated brackets:

$$(8) \quad [X_{i_1}, [X_{i_2}, \ldots, [X_{i_s}, X_{i_{s+1}}] \cdots]], \quad i_1, \ldots, i_s = 1, \ldots, k,\ s = 1, \ldots, r.$$

Thus, in order to check the hypotheses of Theorem 2.7, it suffices to verify if the brackets in (8) evaluated at e_0 are in $\mathcal{D}_{e_0}$, for all $s \geq 1$.

3. The global "single leaf Frobenius Theorem"

3.1. **Sprays on manifolds.** Let M be a smooth manifold and let $\pi : TM \to M$ the canonical projection of its tangent bundle. Denote by $\mathrm{d}\pi : TTM \to TM$ the differential of π; we denote by $\bar{\pi} : TTM \to TM$ the natural projection of $TTM = T(TM)$. For each $a \in \mathbb{R}$ we denote by $\mathfrak{m}_a : TM \to TM$ the operator of multiplication by a.

Definition 3.1. A *spray* on M is a smooth vector field $\mathcal{S} : TM \to TTM$ on the manifold TM satisfying the following two conditions:

(i) $\mathrm{d}\pi \circ \mathcal{S} = \bar{\pi} \circ \mathcal{S}$;
(ii) for all $a \in \mathbb{R}$, $a\,\mathrm{d}\mathfrak{m}_a \circ \mathcal{S} = \mathcal{S} \circ \mathfrak{m}_a$, i.e., $a\,\mathrm{d}\mathfrak{m}_a(v)\mathcal{S}(v) = \mathcal{S}(av)$, for all $v \in TM$.

Remark 3.2. Notice that property (b) on Definition 3.1 implies that a spray vanishes on the zero section of TM. In particular, the integral curves of $\mathcal{S}$ passing through the zero section are constant.

Lemma 3.3. *Let $\mathcal{S} : TM \to TTM$ be a smooth vector field on TM. Then $\mathcal{S}$ is a spray on M if and only if the following conditions are satisfied:*

(a) *for every integral curve $\lambda : I \to TM$ of $\mathcal{S}$, we have $\lambda = \gamma'$, where $\gamma = \pi \circ \lambda$;*

(b) *if $\lambda = \gamma' : I \to TM$ is an integral curve of $\mathcal{S}$ then*

$$I \ni t \longmapsto \frac{\mathrm{d}}{\mathrm{d}t}\gamma(at) \in TM$$

is an integral curve of $\mathcal{S}$, for all $a \in \mathbb{R}$.

Definition 3.4. A curve $\gamma : I \to M$ is called a (maximal) solution of $\mathcal{S}$ if $\gamma' : I \to TM$ is a (maximal) integral curve of the vector field $\mathcal{S}$.

Obviously for every $x \in M$, $v \in T_xM$ there exists a unique maximal solution γ of $\mathcal{S}$ with $\gamma(0) = x$ and $\gamma'(0) = v$.

3.1. **Example** (geodesic spray). If ∇ is a connection on M then one can define a spray $\mathcal{S}$ on M by taking $\mathcal{S}(v)$ to be the unique horizontal vector on T_vTM such that $\mathrm{d}\pi_v\big(\mathcal{S}(v)\big) = v$, for all $v \in TM$. The integral curves of $\mathcal{S}$ are the curves γ', with $\gamma : I \to M$ a geodesic of ∇.

3.2. **Example** (one-parameter subgroup spray). Let G be a Lie group and denote by $\mathfrak{g}$ its Lie algebra. Using left (resp., right) translations, one can identify the tangent bundle TG with the product $G \times \mathfrak{g}$, so that

$$T(TG) \cong T(G \times \mathfrak{g}) \cong (TG) \times (T\mathfrak{g}) \cong (G \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g}).$$

The vector field on TG given by $\mathcal{S}(g, X) = (g, X, X, 0)$, $g \in G$, $X \in \mathfrak{g}$, is a spray in G, whose solutions are left (resp., right) translations of one-parameter subgroups of G. The spray $\mathcal{S}$ is the geodesic spray of the connection whose Christoffel symbols vanish on a left (resp., right) invariant frame.

Let $\mathcal{S}$ be a fixed spray on M and denote by

$$F : \mathrm{Dom}(F) \subset \mathbb{R} \times TM \longrightarrow TM$$

its maximal flow. The *exponential map* associated to $\mathcal{S}$ is the map:

$$\exp(v) = \pi\big(F(1, v)\big) \in M,$$

defined on the set:

$$\mathrm{Dom}(\exp) = \big\{v \in TM : (1, v) \in \mathrm{Dom}(F)\big\}.$$

Since $\mathrm{Dom}(F)$ is open in $\mathbb{R}\times TM$, $\mathrm{Dom}(\exp)$ is open in TM; moreover, by Remark 3.2 the zero section of TM is contained in $\mathrm{Dom}(\exp)$. In particular, for each $x \in M$, the intersection of $\mathrm{Dom}(\exp)$ with T_xM is an open neighborhood of the origin.

Lemma 3.5. *For all $t, s \in \mathbb{R}$, $v \in TM$, $(t, sv) \in \mathbb{R}\times TM$ is in $\mathrm{Dom}(F)$ if and only if $(ts, v) \in \mathbb{R}\times TM$ is in $\mathrm{Dom}(F)$; moreover, $F(t, sv) = sF(ts, v)$.*

Corollary 3.6. *For all $s \in \mathbb{R}$, $v \in TM$, $(s, v) \in \mathbb{R}\times TM$ is in $\mathrm{Dom}(F)$ if and only if sv is in $\mathrm{Dom}(\exp)$; moreover, $\pi\big(F(s,v)\big) = \exp(sv)$.*

Corollary 3.7. *Given $x \in M$, $v \in T_xM$ then the set $\big\{s \in \mathbb{R} : sv \in \mathrm{Dom}(\exp)\big\}$ is an open interval containing the origin; the map $\gamma(s) = \exp(sv)$ defined on such open interval is the maximal solution of $\mathcal{S}$ with $\gamma(0) = x$, $\gamma'(0) = v$.*

For each $x \in M$ let us denote by $\exp_x$ the restriction of exp to $\mathrm{Dom}(\exp)\cap T_xM$. It follows from Corollary 3.7 that the domain of $\exp_x$ is a star-shaped open neighborhood of the origin in T_xM; moreover, $\mathrm{d}\exp_x(0)$ is the identity map of T_xM.

Definition 3.8. A *normal neighborhood* of a point $x \in M$ is an open neighborhood $V \subset M$ of x such that there exists a star-shaped open neighborhood U of the origin in T_xM such that $\exp_x|_U : U \to V$ is a diffeomorphism. An open subset V of M is called *normal*[1] if every $x \in M$ has a normal neighborhood containing V.

It follows from the inverse function theorem that every point of M has a normal neighborhood. Moreover, we have the following:

Proposition 3.9. *Every point of M is contained in some normal open subset of M.*

Proof. Consider the map $\phi : \mathrm{Dom}(\exp) \subset TM \to M \times M$ given by $\phi(v) = \big(\exp(v), \pi(v)\big)$. Given $x \in M$ and denote by $0_x \in TM$ the origin of T_xM. We identify $T_{0_x}TM$ with $T_xM \oplus T_xM$, where the first summand corresponds to the tangent space of the zero section of TM and the second summand corresponds to the tangent space to the fiber of TM containing 0_x. The differential of ϕ at 0_x is easily computed as:

$$\mathrm{d}\phi_{0_x}(v, w) = (v + w, v), \quad v, w \in T_xM.$$

It follows from the inverse function theorem that ϕ carries an open neighborhood $\mathcal{U}$ of 0_x in TM diffeomorphically onto an open neighborhood of

[1]Observe that, according to this definition, a normal open subset of M containing a point $x \in M$ is not necessarily a normal neighborhood of x!

(x, x) in $M \times M$. We can choose $\mathcal{U}$ such that $\mathcal{U} \cap T_yM$ is a star-shaped open neighborhood of the origin of T_yM, for all $y \in \pi(\mathcal{U})$. Let V be an open neighborhood of x in M such that $V \times V \subset \phi(\mathcal{U})$. We claim that V is a normal open subset of M. Let $y \in V$ be fixed. Clearly $V \subset \pi(\mathcal{U})$, so that $\mathcal{U} \cap T_yM$ is a star-shaped open neighborhood of the origin of T_yM; thus $\exp(\mathcal{U} \cap T_yM)$ is a normal neighborhood of y. Moreover, given $z \in V$ then $(z, y) \in V \times V$, so that there exists $v \in \mathcal{U}$ with $\phi(v) = (z, y)$; then $v \in \mathcal{U} \cap T_yM$ and hence $z \in \exp(\mathcal{U} \cap T_yM)$. □

Definition 3.10. A *piecewise solution* of a spray $\mathcal{S}$ is a curve $\gamma : [a, b] \to M$ for which there exists a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$ such that $\gamma|_{[t_i, t_{i+1}]}$ is a solution of $\mathcal{S}$ for all i.

3.2. The global single leaf Frobenius theorem.

Theorem 3.11 (global single leaf Frobenius). *Let E, M be smooth manifolds, $\pi : E \to M$ be a smooth submersion and $\mathcal{D}$ be a smooth horizontal distribution on E. Let $x_0 \in M$, $e_0 \in \pi^{-1}(x_0) \subset E$ be given and let $\mathcal{S}$ be a fixed spray on M. Assume that:*

(a) *every piecewise solution $\gamma : [a, b] \to M$ of $\mathcal{S}$ with $\gamma(a) = x_0$ admits a horizontal lifting $\tilde{\gamma} : [a, b] \to E$ with $\tilde{\gamma}(a) = e_0$;*
(b) *if $\tilde{\gamma} : [a, b] \to E$ is the horizontal lifting of a piecewise solution $\gamma : [a, b] \to M$ of $\mathcal{S}$ with $\tilde{\gamma}(a) = e_0$ then the Levi form of $\mathcal{D}$ vanishes at the point $\tilde{\gamma}(b) \in E$;*
(c) *M is (connected and) simply-connected.*

Then there exists a unique global smooth horizontal section s of E with $s(x_0) = e_0$.

Proof. Uniqueness follows directly from Lemma 2.4. For the existence, we use the globalization theory explained in Appendix A.

Let E' denote the subset of E consisting of the points of the form $\tilde{\gamma}(b)$, where $\tilde{\gamma}(a) = e_0$ and $\tilde{\gamma} : [a, b] \to E$ is the horizontal lifting of some piecewise solution $\gamma : [a, b] \to M$ of $\mathcal{S}$ with $\gamma(a) = x_0$. We define a pre-sheaf $\mathfrak{P}$ over M as follows: for each open subset U of M, $\mathfrak{P}(U)$ is the set of all smooth horizontal sections $s : U \to E$ with $s(U) \subset E'$. Given open subsets $U, V \subset M$ with $V \subset U$ then $\mathfrak{P}_{U,V}$ is given by $\mathfrak{P}_{U,V}(s) = s|_V$, for all $s \in \mathfrak{P}(U)$. The existence of a global smooth horizontal section of E is equivalent to $\mathfrak{P}(M) \neq \emptyset$. We will use Proposition A.8. Using Theorem 2.5 (recall Remark 2.6) we get a smooth horizontal section $s : U \to E$ defined in an open neighborhood U of x_0; it is clear by the construction of s that $s(U) \subset E'$. Thus the pre-sheaf $\mathfrak{P}$ is nontrivial. The localization property (Definition A.4) for $\mathfrak{P}$ is trivial and the uniqueness property (Definition A.6) for $\mathfrak{P}$ follows directly from Lemma 2.4. To conclude the proof, we show that $\mathfrak{P}$ has the extension property (Definition A.6). We shall prove that every normal open subset of M has the extension property for $\mathfrak{P}$ (recall Proposition 3.9). Let U be an open normal subset of M, V be a nonempty open connected subset of U and $s \in \mathfrak{P}(V)$ be a smooth horizontal section of E with $s(V) \subset E'$. Let $x_1 \in V$ be fixed. Since $s(x_1) \in E'$, there exists a piecewise solution $\gamma : [a,b] \to M$ of $\mathcal{S}$ with $\gamma(a) = x_0$ and a horizontal lifting $\tilde\gamma : [a,b] \to E$ of γ with $\tilde\gamma(a) = e_0$ and $\tilde\gamma(b) = s(x_1)$. Let W be a normal neighborhood of x_1 containing U and W_0 be a star-shaped open neighborhood of the origin in $T_{x_1}M$ such that $\exp_{x_1} : W_0 \to W$ is a diffeomorphism. For each $x \in W$ let $v \in W_0$ be such that $\exp_{x_1}(v) = x$; we claim that $\mu_x : [0,1] \ni t \mapsto \exp_{x_1}(tv) \in M$ has a horizontal lifting $\tilde\mu : [0,1] \to E$ starting at $s(x_1)$ and that the Levi form of $\mathcal{D}$ vanishes along the image of $\tilde\mu$. Namely, the concatenation $\gamma \cdot \mu$ of γ with μ is a piecewise solution of $\mathcal{S}$ starting at x_0; by hypothesis (a), $\gamma \cdot \mu$ has a horizontal lifting starting at e_0. Such horizontal lifting is of the form $\tilde\gamma \cdot \tilde\mu$, where $\tilde\mu$ is a horizontal lifting of μ starting at $s(x_1)$; moreover, hypothesis (b) implies that the Levi form of $\mathcal{D}$ vanishes along $\tilde\gamma \cdot \tilde\mu$. Observe that the image of $\tilde\mu$ is contained in E'. We can now apply Theorem 2.5 to obtain a smooth horizontal section $\bar s : W \to E$ with $\bar s(x_1) = s(x_1)$. Thus, by Lemma 2.4 and the connectedness of V, $\bar s|_V = s$ and hence $\bar s|_U \in \mathfrak{P}(U)$ is an extension of s to U. □

Proposition 3.12. *Let E, M be real-analytic manifolds, $\pi : E \to M$ be a real-analytic submersion and $\mathcal{D}$ be a real-analytic horizontal distribution on E. Assume that:*

(a) *M is (connected and) simply-connected;*
(b) *given a real analytic curve $\gamma : I \to M$, $t_0 \in I$ and $e_0 \in \pi^{-1}(\gamma(t_0))$ then there exists a horizontal lifting $\tilde\gamma : I \to E$ of γ with $\tilde\gamma(t_0) = e_0$.*

Then any local horizontal section $s : U \to E$ of π defined on a nonempty connected open subset U of M extends to a global horizontal section of π. In particular, if $\mathcal{D}$ satisfies the hypothesis of Theorem 2.7 at some point e_0 of E, assumptions (a) and (b) imply that π admits a global horizontal section.

Proof. We use again the globalization theory explained in Appendix A. We define a pre-sheaf $\mathfrak{P}$ over M as follows: for each open subset U of M, $\mathfrak{P}(U)$ is the set of all smooth horizontal sections $s : U \to E$; given open subsets $U, V \subset M$ with $V \subset U$ then $\mathfrak{P}_{U,V}$ is given by $\mathfrak{P}_{U,V}(s) = s|_V$, for all $s \in \mathfrak{P}(U)$. By Proposition A.8 it suffices to show that $\mathfrak{P}$ has the localization property, the uniqueness property and the extension property. The localization property is trivial and the uniqueness property follows from Lemma 2.4. As to the extension property, it can be proved as follows.

Let $x_0 \in M$ be fixed and let $\varphi : U \to \mathrm{B}_0(r)$ be a real-analytic chart defined on an open neighborhood U of x_0, taking values in the open ball $\mathrm{B}_0(r) \subset \mathbb{R}^n$ of radius r centered at the origin and $\varphi(x_0) = 0$. We will show that $V = \varphi^{-1}\big(\mathrm{B}_0(r/3)\big)$ is an open neighborhood of x_0 having the extension property for $\mathfrak{P}$. To this aim, let W be a nonempty connected open subset of V and let $s \in \mathfrak{P}(W)$ be a local horizontal section defined on W. Choose $x_1 \in W$. Set $\Lambda = \mathrm{B}_0\big(\frac{2}{3}r\big)$ and let $\psi : Z \subset \mathbb{R} \times \Lambda \to M$ be the one-parameter family of curves defined by $\psi(t,\lambda) = \varphi^{-1}\big(\varphi(x_1) + t\lambda\big)$, where Z is the set of pairs $(t,\lambda) \in \mathbb{R} \times \Lambda$ with $\varphi(x_1) + t\lambda \in \mathrm{B}_0(r)$. We define a local right inverse

$$\alpha : \varphi^{-1}\big(\mathrm{B}_{\varphi(x_1)}(\tfrac{2}{3}r)\big) \longrightarrow Z \subset \mathbb{R} \times \Lambda$$

of ψ by setting $\alpha(x) = \big(1, \varphi(x) - \varphi(x_1)\big)$. By assumption (b), for each $\lambda \in \Lambda$, the curve $t \mapsto \psi(t,\lambda)$ has a horizontal lifting $t \mapsto \tilde{\psi}(t,\lambda) \in E$ with $\tilde{\psi}(0,\lambda) = s(x_1)$. Notice that, by the uniqueness of the horizontal lifting of a curve, we have $\tilde{\psi}(t,\lambda) = s\big(\psi(t,\lambda)\big)$ for small t. Since s is a horizontal section of π, its image is an integral submanifold of $\mathcal{D}$ and thus the Levi form $\mathfrak{L}^{\mathcal{D}}$ vanishes along the image of s. Thus $\mathfrak{L}^{\mathcal{D}}$ vanishes at the point $\tilde{\psi}(t,\lambda)$ for small t; hence, since $t \mapsto \mathfrak{L}^{\mathcal{D}}\big(\tilde{\psi}(t,\lambda)\big)$ is real-analytic, $\mathfrak{L}^{\mathcal{D}}$ must vanish along the entire curve $t \mapsto \tilde{\psi}(t,\lambda)$. By Theorem 2.5, $\tilde{\psi} \circ \alpha$ is a horizontal section of E with $(\tilde{\psi} \circ \alpha)(x_1) = s(x_1)$; since the domain of α clearly contains V, Lemma 2.4 implies that $\tilde{\psi} \circ \alpha$ extends s to (an open set containing) V. This proves the extension property of $\mathfrak{P}$ and concludes the proof. □

4. Levi–Civita connections

4.1. **Levi form of the horizontal distribution of a connection.** Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M and let ∇ be a connection on E; for $m \in M$ we denote by $E_m = \pi^{-1}(m)$ the fiber of E over m. We denote by $R_m : T_mM \times T_mM \times E_m \to E_m$ the *curvature tensor* of ∇ defined by:

$$R(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi,$$

for all smooth vector fields X, Y in M and every smooth section ξ of E.

Recall that there exists a unique distribution $\mathcal{D}$ on the manifold E that is horizontal with respect to π and has the following property: if $\gamma : I \to M$ is a smooth curve on M then a curve $\tilde{\gamma} : I \to E$ is a horizontal lifting of γ with respect to $\mathcal{D}$ if and only if $\tilde{\gamma}$ is a ∇-parallel section of E along γ. We call $\mathcal{D}$ the *horizontal distribution* of ∇. Given $m \in M$ and $\xi \in E_m$ then the quotient $T_\xi E/\mathcal{D}_\xi$ can be identified with $T_\xi(E_m) = \mathrm{Ker}(\mathrm{d}\pi_\xi)$; moreover, since E_m is a vector space, we identify $T_\xi(E_m)$ with E_m. We also identify $\mathcal{D}_\xi$ with T_mM using $\mathrm{d}\pi_\xi$. The Levi form of $\mathcal{D}$ at a point $\xi \in E$ can thus be seen as a bilinear map:

$$\mathfrak{L}^{\mathcal{D}}_\xi : T_mM \times T_mM \longrightarrow E_m.$$

Lemma 4.1. *The Levi form of the horizontal distribution $\mathcal{D}$ of a connection ∇ is given by:*

$$\mathfrak{L}^{\mathcal{D}}_\xi(v,w) = -R_m(v,w)\xi,$$

for all $m \in M$, $\xi \in E_m$.

Proof. Given a smooth vector field X on M we denote by X^{hor} the *horizontal lift* of X which is the unique horizontal vector field on E such that $\mathrm{d}\pi_\xi(X^{\mathrm{hor}}(\xi)) = X\big(\pi(\xi)\big)$, for all $\xi \in E$. Given smooth vector fields X, Y on M, we have to show that vertical component of $[X^{\mathrm{hor}}, Y^{\mathrm{hor}}]$ at a point $\xi \in E$ is equal to $-R(X,Y)\xi$. Note that the horizontal component of $[X^{\mathrm{hor}}, Y^{\mathrm{hor}}]$ is $[X,Y]^{\mathrm{hor}}$, since X^{hor} and Y^{hor} are π-related respectively with X and Y. Thus, the proof will be concluded once we show that:

$$\alpha\big([X^{\mathrm{hor}}, Y^{\mathrm{hor}}] - [X,Y]^{\mathrm{hor}}\big) = -\alpha\big(R(X,Y)\xi\big),$$

for every smooth section α of the dual bundle E^*. Given one such section α, we denote by $f_\alpha : E \to \mathbb{R}$ the smooth map defined by:

$$f_\alpha(\xi) = \alpha(\xi).$$

We claim that:

$$X^{\mathrm{hor}}(f_\alpha) = f_{\nabla^*_X\alpha},$$

where ∇^* denotes the connection of E^*. Namely, let $\gamma : \left]-\varepsilon, \varepsilon\right[\to M$ be an integral curve of X and let $t \mapsto \xi(t)$ be a parallel section of E along γ, so that ξ is an integral curve of X^{hor}; then:

$$\begin{aligned} X^{\mathrm{hor}}(f_\alpha) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f_\alpha\big(\xi(t)\big) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \alpha_{\gamma(t)}\big(\xi(t)\big) \\ = \big(\nabla^*_{\gamma'(t)}\alpha\big)\xi(t)\big|_{t=0} = (\nabla^*_X\alpha)\xi, \end{aligned}$$

which proves the claim. Observe also that if $v \in TE$ is a vertical vector then $v(f_\alpha) = \alpha(v)$; therefore:

(9)
$$\alpha\big([X^{\mathrm{hor}}, Y^{\mathrm{hor}}] - [X,Y]^{\mathrm{hor}}\big) = \big([X^{\mathrm{hor}}, Y^{\mathrm{hor}}] - [X,Y]^{\mathrm{hor}}\big)(f_\alpha) = f_{R^*(X,Y)\alpha},$$

where R^* denotes the curvature tensor of ∇^*. A simple computation shows that:

$$R^*(X,Y)\alpha = -\alpha \circ R(X,Y).$$

The conclusion follows from (9) by evaluating both sides at the point ξ. $\square$

Corollary 4.2. *Let $\pi : E \to M$ be a smooth vector bundle endowed with a connection ∇, let $\psi : Z \subset \mathbb{R} \times \Lambda \to M$ be a Λ-parametric family of curves on M with a local right inverse $\alpha : V \subset M \to Z$ and let $\tilde{\psi} : Z \to E$ be a smooth section of E along ψ such that $t \mapsto \tilde{\psi}(t, \lambda)$ is parallel for all $\lambda \in \Lambda$ and such that $\lambda \mapsto \tilde{\psi}(0, \lambda)$ is also parallel. If*

$$R_{\psi(t,\lambda)}(v, w)\tilde{\psi}(t, \lambda) = 0,$$

for all $v, w \in T_{\psi(t,\lambda)}M$ and all $(t, \lambda) \in Z$ then $s = \tilde{\psi} \circ \alpha$ is a parallel local section of E.

Proof. Follows readily from Theorem 2.5 and Lemma 4.1. □

Corollary 4.3. *Let $\pi : E \to M$ be a smooth vector bundle endowed with a connection ∇. Let $x_0 \in M$, $e_0 \in \pi^{-1}(x_0) \subset E$ be given and let $\mathcal{S}$ be a fixed spray on M. Assume that:*

(a) *if $\gamma : [a, b] \to M$ is a piecewise solution of $\mathcal{S}$ with $\gamma(a) = x_0$ and $\tilde{\gamma} : [a, b] \to E$ is a parallel section of E along γ with $\tilde{\gamma}(a) = e_0$ then $R_{\gamma(b)}(v, w)\tilde{\gamma}(b) = 0$, for all $v, w \in T_{\gamma(b)}M$;*

(b) *M is (connected and) simply-connected.*

Then there exists a unique global smooth parallel section s of E with $s(x_0) = e_0$.

Proof. Follows directly from Lemma 4.1 and Theorem 3.11. □

Corollary 4.4. *Let $\pi : E \to M$ be a real-analytic vector bundle endowed with a real-analytic connection ∇. Assume that M is (connected and) simply-connected. Then any local parallel section $s : U \to E$ of E defined on a nonempty connected open subset U of M extends to a global parallel section of E.*

Proof. It follows from Lemma 4.1 and Proposition 3.12. □

Proposition 4.5. *Let $\pi : E \to M$ be a real-analytic vector bundle endowed with a real-analytic connection ∇. Given $x \in M$, $e \in \pi^{-1}(x)$, assume that:*

$$(\nabla^k R)(v_1, v_2, \ldots, v_{k+2})e = 0, \tag{10}$$

for all $v_1, \ldots, v_{k+2} \in T_xM$ and all $k \geq 0$. Then there exists a parallel section s of E defined in an open neighborhood of x in M with $s(x) = e$; in particular, by Corollary 4.4, if M is (connected and) simply-connected then there exists a global parallel section s of E with $s(x) = e$.

Proof. Given a smooth vector field X on M, we denote by $\widehat{X}$ the unique horizontal vector field on E that is π-related with X. We show that condition (10) is equivalent to the condition that all iterated brackets of vector fields $\widehat{X}$ are horizontal at the point e. The conclusion will then follow from

Theorem 2.7. First, let us compute the bracket $[\widehat{X},\widehat{Y}]$. Since $\widehat{X}$ and $\widehat{Y}$ are π-related respectively with X and Y, it follows that the horizontal component of $[\widehat{X},\widehat{Y}]$ is $[X,Y]$; its vertical component is computed in Lemma 4.1. Thus:

$$[\widehat{X},\widehat{Y}]_e = \big([X,Y]_x, -R(X,Y)e\big), \tag{11}$$

where we write tangent vectors to E as pairs consisting of a horizontal component and a vertical component. Given a smooth section L of the vector bundle $\mathrm{Lin}(E)$, we denote by $\widetilde{L}$ the vertical vector field on E defined by $\widetilde{L}(e) = \big(0, L(e)\big)$. Given a smooth vector field Z on M, let us compute the bracket $[\widehat{Z},\widetilde{L}]$. Since $\widehat{Z}$ is π-related with Z and $\widetilde{L}$ is π-related with zero, it follows that $[\widehat{Z},\widetilde{L}]$ is vertical. Given a smooth section α of E^*, we consider the map $f_\alpha : E \to \mathbb{R}$ defined by $f_\alpha(e) = \alpha(e)$ and we compute as follows:

$$\widetilde{L}(f_\alpha)(e) = \alpha\big(L(e)\big) = f_{\alpha\circ L}(e),$$
$$\widehat{Z}(f_\alpha)(e) = \frac{\mathrm{d}}{\mathrm{d}t} f_\alpha\big(e(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\,\alpha\big(e(t)\big) = (\nabla_Z\alpha)(e) = f_{\nabla_Z\alpha}(e),$$

where $t \mapsto e(t)$ is an integral curve of $\widehat{Z}$, i.e., a parallel section of E along an integral curve of Z. Then:

$$\begin{aligned}[\widehat{Z},\widetilde{L}](f_\alpha) = \widehat{Z}\big(\widetilde{L}(f_\alpha)\big) - \widetilde{L}\big(\widehat{Z}(f_\alpha)\big) = f_{\nabla_Z(\alpha\circ L)} - f_{(\nabla_Z\alpha)\circ L} &= f_{\alpha\circ\nabla_Z L}\\ &= \widetilde{\nabla_Z L}(f_\alpha),\end{aligned}$$

so that:

$$[\widehat{Z},\widetilde{L}] = \widetilde{\nabla_Z L}. \tag{12}$$

Notice that (11) says that $[\widehat{X},\widehat{Y}]$ is given by:

$$[\widehat{X},\widehat{Y}] = \widehat{[X,Y]} - \widetilde{L},$$

where $L(e) = R(X,Y)e$. Using the equality above and (12) it can be easily proved by induction that:

$$[\widehat{Z}_1,[\widehat{Z}_2,\ldots[\widehat{Z}_k,[\widehat{X},\widehat{Y}]]\cdots]] = [\widehat{Z}_1,[\widehat{Z}_2,\ldots[\widehat{Z}_k,\widehat{[X,Y]}]\cdots]] - \widetilde{L_k},$$

where:

$$L_k(e) = (\nabla_{Z_1}(\nabla_{Z_2}(\cdots\nabla_{Z_k}(R(X,Y))\cdots)))e.$$

The conclusion follows by observing that $L_k(e)$ can be written in the form:

$$L_k(e) = (\nabla^k R)(Z_1,\ldots,Z_k,X,Y)e + \sum_{i=0}^{k-1} L_{ki},$$

where L_{ki} is a term linear in $(\nabla^i R)(\cdots)e$. $\square$

4.2. **Connections arising from metric tensors.** Let $\pi : E \to M$ be a vector bundle and let $E^* \otimes E^*$ denote the vector bundle over M whose fiber at $m \in M$ is the space of bilinear forms on E_m. If ∇ is a connection on E then we can define a induced connection ∇^{bil} on $E^* \otimes E^*$ by setting:

$$(\nabla^{\mathrm{bil}}_X g)(\xi, \eta) = X\big(g(\xi, \eta)\big) - g(\nabla_X \xi, \eta) - g(\xi, \nabla_X \eta),$$

where X is a smooth vector field on M and ξ, η are smooth sections of E. A straightforward computation shows that the curvature tensor R^{bil} of ∇^{bil} is given by:

$$(13) \qquad \big(R^{\mathrm{bil}}(X, Y)g\big)(\xi, \eta) = -g\big(R(X, Y)\xi, \eta\big) - g\big(\xi, R(X, Y)\eta\big),$$

for any smooth vector fields X, Y on M, any smooth sections ξ, η of E and any smooth section g of $E^* \otimes E^*$. If $\gamma : I \to M$ is a smooth curve defined on an interval I around 0 and if g_0 is a bilinear form on $E_{\gamma(0)}$ then the parallel transport $I \ni t \mapsto g_t$ of g_0 along γ relatively to the connection ∇^{bil} is given by:

$$g_t(\xi, \eta) = g_0(P_t^{-1}\xi, P_t^{-1}\eta), \quad \xi, \eta \in E_{\gamma(t)},$$

where $P_t : E_{\gamma(0)} \to E_{\gamma(t)}$ denotes the parallel transport along γ.

Given a smooth manifold M then a *semi-Riemannian metric* on M is a smooth section g of the vector bundle $TM^* \otimes TM^*$ such that $g_m : T_mM \times T_mM \to \mathbb{R}$ is symmetric and nondegenerate; if g_m is positive definite for all $m \in M$, we call g a *Riemannian metric.* The *Levi-Civita connection* of g is the unique symmetric connection ∇ on TM such that $\nabla^{\mathrm{bil}} g = 0$.

We consider the following problem: *given a symmetric connection ∇ on a smooth manifold M, when does there exist a semi-Riemannian metric g on M such that ∇ is the Levi-Civita connection of g?*

Note that if ∇ is the Levi-Civita connection of a semi-Riemannian metric g then for any $m \in M$ and any $v, w \in T_mM$, the linear operator $R_m(v, w) : T_mM \to T_mM$ corresponding to the curvature tensor of ∇ is anti-symmetric with respect to g_m; moreover, given a smooth curve $\gamma : [a, b] \to M$ with $\gamma(a) = m_0$ and $\gamma(b) = m$ then, denoting by $P : T_{m_0}M \to T_mM$ the parallel transport along γ, the linear operator:

$$P^{-1}\big[R_m(v, w)\big]P : T_{m_0}M \longrightarrow T_{m_0}M$$

is anti-symmetric with respect to g_{m_0}, for all $v, w \in T_mM$. We will show below that this anti-symmetry characterizes the connections arising from semi-Riemannian metrics.

Proposition 4.6. *Let M be a smooth manifold, ∇ be a symmetric connection on TM, $m_0 \in M$ and g_0 be a nondegenerate symmetric bilinear form on $T_{m_0}M$. Let $\psi : Z \subset \mathbb{R} \times \Lambda \to M$ be a Λ-parametric family of curves on*

M with a local right inverse $\alpha : V \subset M \to Z$; assume that $\psi(0,\lambda) = m_0$, for all $\lambda \in M$. For each $(t,\lambda) \in Z$, we denote by $P_{(t,\lambda)} : T_{m_0}M \to T_{\psi(t,\lambda)}M$ the parallel transport along $t \mapsto \psi(t,\lambda)$. Assume that for all $(t,\lambda) \in Z$ the linear operator:

$$P_{(t,\lambda)}^{-1}\big[R_{\psi(t,\lambda)}(v,w)\big]P_{(t,\lambda)} : T_{m_0}M \longrightarrow T_{m_0}M \tag{14}$$

is anti-symmetric with respect to g_0, for all $v, w \in T_{\psi(t,\lambda)}M$, where

$$R_{\psi(t,\lambda)}(v,w) : T_{\psi(t,\lambda)}M \longrightarrow T_{\psi(t,\lambda)}M$$

denotes the linear operator corresponding to the curvature tensor of ∇. Then ∇ is the Levi-Civita connection of the semi-Riemannian metric g on $V \subset M$ defined by setting:

$$g_m(\cdot,\cdot) = g_0(P_{\alpha(m)}^{-1}\cdot, P_{\alpha(m)}^{-1}\cdot),$$

for all $m \in V$.

Proof. For each $(t,\lambda) \in Z$, let $\tilde{\psi}(t,\lambda) \in TM^* \otimes TM^*$ be the bilinear form on $T_{\psi(t,\lambda)}M$ defined by:

$$\tilde{\psi}(t,\lambda)(\cdot,\cdot) = g_0(P_{(t,\lambda)}^{-1}\cdot, P_{(t,\lambda)}^{-1}\cdot).$$

Then $\tilde{\psi}$ satisfies the hypotheses of Corollary 4.2 with $E = TM^* \otimes TM^*$; namely, $\tilde{\psi}(0,\lambda) = g_0$, for all $\lambda \in \Lambda$ and by (13) and the anti-symmetry of (14), we have $R^{\mathrm{bil}}_{\psi(t,\lambda)}(v,w) = 0$, for all $v, w \in T_{\phi(t,\lambda)}M$. Hence $g = \tilde{\psi} \circ \alpha : V \to TM^* \otimes TM^*$ is a parallel section of $TM^* \otimes TM^*$ and ∇ is the Levi-Civita connection of g. □

Theorem 4.7. *Let M be a smooth manifold, ∇ be a symmetric connection on TM, $m_0 \in M$ and g_0 be a nondegenerate symmetric bilinear form on $T_{m_0}M$. Let $\mathcal{S}$ be a fixed spray on M. Assume that:*

- *for every piecewise solution $\gamma : [a,b] \to M$ of $\mathcal{S}$ with $\gamma(a) = m_0$ the linear operator $P_\gamma^{-1}R_{\gamma(b)}P_\gamma$ on $T_{m_0}M$ is g_0-anti-symmetric, where $P_\gamma : T_{m_0}M \to T_{\gamma(b)}M$ denotes parallel transport along γ;*
- *M is (connected and) simply-connected.*

Then g_0 extends to a semi-Riemannian metric on M for which ∇ is the Levi-Civita connection.

Proof. It follows from (13) and Corollary 4.3. □

Proposition 4.8. *Let M be a (connected and) simply-connected real-analytic manifold and let ∇ be a real-analytic symmetric connection on TM. If there exists a semi-Riemannian metric g on a nonempty open connected subset of M having ∇ as its Levi-Civita connection then g extends to a* *globally defined semi-Riemannian metric on M having ∇ as its Levi-Civita connection.*

Proof. It follows from Corollary 4.4. □

Proposition 4.9. *Let M be a real-analytic manifold and let ∇ be a real-analytic symmetric connection on TM. Given a point $x_0 \in M$ and a nondegenerate symmetric bilinear form g_0 on $T_{x_0}M$, if:*

$$(\nabla^k R)(v_1, \ldots, v_{k+2}) : T_{x_0}M \to T_{x_0}M$$

is g_0-anti-symmetric for all $v_1, \ldots, v_{k+2} \in T_{x_0}M$ and all $k \geq 0$ then g_0 extends to a semi-Riemannian metric on an open neighborhood of x_0 whose Levi-Civita connection is ∇. Moreover, if M is (connected and) simply-connected then g_0 extends to a global semi-Riemannian metric on M having ∇ as its Levi-Civita connection.

Proof. Follows easily from Proposition 4.5 and from formula (13). □

The above characterizations of Levi–Civita connections have been used in [6], where the authors study left-invariant (symmetric) connections in Lie groups.

5. Affine maps

Let us now discuss as an application of the "single leaf Frobenius Theorem" a classical result in differential geometry.

5.1. **The Cartan–Ambrose–Hicks Theorem.** Consider the following setup. Let M, N be smooth manifolds endowed respectively with connections ∇^M and ∇^N. We denote by T^M, T^N (resp., R^M, R^N) respectively the torsion tensors (resp., curvature tensors) of ∇^M and ∇^N. A smooth map $f : M \to N$ is called *affine* if for every $x \in M$, $v \in T_xM$ and every smooth vector field X on M we have:

$$\mathrm{d}f_x(\nabla^M_v X) = \nabla^N_v(\mathrm{d}f \circ X);$$

in the formula above $\mathrm{d}f \circ X : M \to TN$ is regarded as a vector field along f on N, so that it makes sense to compute its covariant derivative ∇^N along $v \in TM$.

Let $x_0 \in M$, $y_0 \in N$ be given and let $\sigma_0 : T_{x_0}M \to T_{y_0}N$ be a linear map. Given a geodesic $\gamma : [a,b] \to M$ with $\gamma(a) = x_0$ then the geodesic $\mu : [a,b] \to N$ with $\mu(a) = y_0$ and $\mu'(a) = \sigma\big(\gamma'(a)\big)$ is called *induced* on N by the geodesic γ and by σ_0. We observe that the geodesic $\mu : [a,b] \to N$ is well-defined only if $(b-a)\sigma\big(\gamma'(a)\big)$ is in the domain of the exponential map of N at the point y_0. Let $\sigma : T_{\gamma(b)}M \to T_{\mu(b)}N$ be the linear map given by the composition of parallel transport along γ, σ_0 and parallel transport along μ; we call σ the linear map *induced* by γ and σ_0.

Theorem 5.1. *Let $x_0 \in M$, $y_0 \in N$ be given and let $\sigma_0 : T_{x_0}M \to T_{y_0}N$ be a linear map. Let U be an open subset of $T_{x_0}M$ which is star-shaped at the origin and which is carried diffeomorphically onto an open subset V of M by the exponential map of M at x_0. Assume that $\sigma(U)$ is contained in the domain of the exponential map of N at y_0. For each $x \in V$, let $\gamma_x : [0,1] \to M$ be the unique geodesic such that $\gamma_x'(0) \in U$ and $\gamma_x(1) = x$; let $\mu_x : [0,1] \to N$ and $\sigma_x : T_xM \to T_{\mu_x(1)}N$ be respectively the geodesic and the linear map induced by γ_x and σ_0. Assume that for all $x \in V$ the linear map σ_x relates T^M with T^N and R^M with R^N, i.e.:*

$$\sigma_x\big(T^M(\cdot,\cdot)\big) = T^N\big(\sigma_x(\cdot),\sigma_x(\cdot)\big), \quad \sigma_x\big(R^M(\cdot,\cdot)\,\cdot\,\big) = R^N\big(\sigma_x(\cdot),\sigma_x(\cdot)\big)\sigma_x(\cdot).$$

Then the smooth map $f : V \to N$ defined by $f(x) = \mu_x(1)$ is affine and $\mathrm{d}f(x) = \sigma_x$ for all $x \in V$; in particular, $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.

Remark 5.2. In the statement of Theorem 5.1, if one assumes that σ_0 is an isomorphism (resp., injective) then it follows that f is a local diffeomorphism (resp., that f is an immersion). Moreover, if ∇^M and ∇^N are the Levi-Civita connections of Riemannian metrics on M and N respectively then, if one assumes that σ_0 is an isometry, it follows that f is a local isometry.

In what follows we assume that ∇^N is *geodesically complete*, i.e., for all $y \in N$ the exponential map of N at y is defined on the whole tangent space T_yN.

Let $x_0 \in M$, $y_0 \in N$ be given and let $\sigma_0 : T_{x_0}M \to T_{y_0}N$ be a linear map. Let $\gamma : [a,b] \to M$ be a *piecewise geodesic* with $\gamma(a) = x_0$, i.e., there exists a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a,b]$ such that $\gamma|_{[t_i,t_{i+1}]}$ is a geodesic for all i. Using the linear map σ_0 it is possible to define a piecewise geodesic $\mu : [a,b] \to N$ and a linear map $\sigma : T_{\gamma(b)}M \to T_{\mu(b)}N$ *induced* by γ in the following way: we first define inductively a sequence of geodesics $\mu_i : [t_i, t_{i+1}] \to N$ and of linear maps $\sigma_i : T_{\gamma(t_i)}M \to T_{\mu_i(t_i)}N$. Let μ_0 and σ_1 be respectively the geodesic and the linear map induced by the geodesic $\gamma|_{[t_0,t_1]}$ and by σ_0. Assuming that μ_i and σ_{i+1} are defined we let μ_{i+1} and σ_{i+2} be respectively the geodesic and the linear map induced by the geodesic $\gamma|_{[t_i,t_{i+1}]}$ and by σ_{i+1}. Finally, we let $\mu : [a,b] \to N$ be the piecewise geodesic such that $\mu|_{[t_i,t_{i+1}]} = \mu_i$ for all i and we let $\sigma = \sigma_k$.

Theorem 5.3 (Cartan–Ambrose–Hicks). *Let M, N be smooth manifolds endowed respectively with connections ∇^M and ∇^N; assume that ∇^N is geodesically complete and that M is connected and simply-connected. Let $x_0 \in M$, $y_0 \in N$ be given and let $\sigma_0 : T_{x_0}M \to T_{y_0}N$ be a linear map.*

For each piecewise geodesic $\gamma : [a,b] \to M$ with $\gamma(a) = x_0$ denote by $\mu_\gamma : [a,b] \to N$ and by $\sigma_\gamma : T_{\gamma(b)}M \to T_{\mu_\gamma(b)}N$ respectively the piecewise geodesic and the linear map induced by the piecewise geodesic γ and by σ_0. Assume that for every piecewise geodesic γ the linear map σ_γ relates T^M with T^N and R^M with R^N. Then there exists a smooth affine map $f : M \to N$ such that for every piecewise geodesic $\gamma : [a,b] \to M$ we have $f \circ \gamma = \mu_\gamma$ and $\mathrm{d}f\big(\gamma(b)\big) = \sigma_\gamma$; in particular, $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.

Remark 5.4. In the statement of the Cartan–Ambrose–Hicks Theorem, if one assumes in addition that σ_0 is an isomorphism, and that ∇^M is geodesically complete then it follows that the affine map $f : M \to N$ is a covering map.

Corollary 5.5. *Let (M, g^M), (N, g^N) be Riemannian manifolds with (N, g^N) complete and M connected and simply-connected. Let $x_0 \in M$, $y_0 \in N$ be given and let $\sigma_0 : T_{x_0}M \to T_{y_0}N$ be a linear isometry onto a subspace of $T_{y_0}N$. For each piecewise geodesic $\gamma : [a,b] \to M$ with $\gamma(a) = x_0$ denote by $\mu_\gamma : [a,b] \to N$ and by $\sigma_\gamma : T_{\gamma(b)}M \to T_{\mu_\gamma(b)}N$ respectively the piecewise geodesic and the linear map induced by the piecewise geodesic γ and by σ_0. Assume that for every piecewise geodesic γ the linear map σ_γ relates R^M with R^N. Then there exists a totally geodesic isometric immersion $f : M \to N$ with $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.*

Proof. It follows immediately from Theorem 5.3; observe that the condition that f is totally geodesic follows from the fact that f is affine. □

We now show how the proof of Theorems 5.1 and 5.3 can be obtained as an application of the local and the global version of the "single leaf Frobenius Theorem" (Theorems 2.5 and 3.11).

Consider the vector bundle $E = \mathrm{Lin}(TM, TN)$ over $M \times N$ whose fiber at a point $(x,y) \in M \times N$ is the space of linear maps $\mathrm{Lin}(T_xM, T_yN)$. Notice that E coincides with the tensor bundle $\pi_1^*(TM^*) \otimes \pi_2^*(TN)$, where π_1 and π_2 denote the projections of the product $M \times N$. The connections ∇^M and ∇^N naturally induce a connection ∇ on E given by:

$$(\nabla_{(v,w)}\sigma)(X) = \nabla^N_{(v,w)}\big(\sigma(X)\big) - \sigma(\nabla^M_v X), \tag{15}$$

where $v \in TM$, $w \in TN$, X is a smooth vector field on M and $\sigma : M \times N \to E$ is a smooth section of E. In the formula above, $\sigma(X) : M \times N \to TN$ is regarded as vector field along the projection $\pi_2 : M \times N \to N$ on N.

Given a smooth map $f : U \to N$ defined on an open subset U of M then the differential $\mathrm{d}f : U \to E$ can be regarded as section of E along the map $U \ni x \mapsto \big(x, f(x)\big) \in M \times N$, so that it makes sense to consider the covariant derivative of $\mathrm{d}f$ with respect to the connection ∇.

Lemma 5.6. *A smooth map $f : U \to N$ defined on an open subset of M is affine if and only if* $\mathrm{d}f$ *is parallel with respect to* ∇.

Proof. Given $v \in TM$ and a smooth vector field X on U we compute:

$$\big(\nabla_v(\mathrm{d}f)\big)(X) = \nabla^N_v\big(\mathrm{d}f(X)\big) - \mathrm{d}f(\nabla^M_v X).$$

The conclusion follows. □

Lemma 5.7. *Let $\lambda : t \mapsto \big(\gamma(t), \mu(t), \sigma(t)\big)$ be a smooth curve on E, i.e., γ is a curve on M, μ is a curve on N and $\sigma(t)$ is a linear map from $T_{\gamma(t)}M$ to $T_{\mu(t)}N$ for all t. Then λ is parallel with respect to ∇ (or, equivalently, λ is tangent to the horizontal distribution corresponding to ∇) if and only if the following condition holds: for every ∇^M-parallel vector field $t \mapsto v(t) \in TM$ along γ, the vector field $t \mapsto \sigma(t)v(t) \in TN$ along μ is ∇^N-parallel.*

Proof. Let $t \mapsto v(t)$ be a vector field along γ. Let us denote by $\frac{\mathrm{D}}{\mathrm{d}t}$, $\frac{\mathrm{D}^M}{\mathrm{d}t}$ and $\frac{\mathrm{D}^N}{\mathrm{d}t}$ respectively the covariant derivatives with respect to the parameter t corresponding to the connections ∇, ∇^M and ∇^N. The conclusion follows easily from the following formula:

$$\tfrac{\mathrm{D}^N}{\mathrm{d}t}[\sigma(t)v(t)] = \Big(\tfrac{\mathrm{D}}{\mathrm{d}t}\,\sigma(t)\Big)v(t) + \sigma(t)\tfrac{\mathrm{D}^M}{\mathrm{d}t}\,v(t),$$

observing that λ is ∇-parallel if and only if $\frac{\mathrm{D}}{\mathrm{d}t}\sigma(t) = 0$. □

The geometric interpretation of Lemma 5.7 is given by the following:

Corollary 5.8. *Let λ be as in the statement of Lemma 5.7 and let t_0 in the domain of λ be fixed. Then λ is parallel with respect to ∇ if and only if the following condition holds: for all t, the linear map $\sigma(t) : T_{\gamma(t)}M \to T_{\mu(t)}N$ is given by the composition of ∇^M-parallel transport along γ, $\sigma(t_0)$ and ∇^N-parallel transport along μ.* □

We now explain in which form the "single leaf Frobenius Theorem" (Theorem 2.5) is going to be applied. We consider the smooth submersion $\pi : E \to M$ given by the composition of the canonical projection $E \to M \times N$ with the first projection $\pi_1 : M \times N \to M$. Given $x \in M$, $y \in N$, $\sigma \in \mathrm{Lin}(T_xM, T_yN)$ then the tangent space $T_\sigma E$ is identified with the direct sum of $T_xM \oplus T_yN$ (the horizontal space corresponding to the connection ∇) and $\mathrm{Lin}(T_xM, T_yN)$ (the tangent space to the fiber). We will now define a distribution $\mathcal{D}$ on the manifold E that is horizontal with respect to the submersion $\pi : E \to M$. We set:

$$\mathcal{D}_\sigma = \mathrm{Gr}(\sigma) \oplus \{0\} \subset T_xM \oplus T_yN \oplus \mathrm{Lin}(T_xM, T_yN) \cong T_\sigma E, \tag{16}$$

where $\mathrm{Gr}(\sigma) \subset T_xM \oplus T_yN$ denotes the graph of the linear map σ.

Lemma 5.9. *Let $s : U \to E$ be a smooth section of E defined on an open subset U of M; we write $s(x) = \big(f(x), \sigma(x)\big)$, where $f : U \to N$ is a smooth map and $\sigma(x) \in \mathrm{Lin}(T_xM, T_{f(x)}N)$, for all $x \in U$. Then s is $\mathcal{D}$-horizontal if and only if $\sigma(x) = \mathrm{d}f(x)$ for all $x \in U$ and f is affine.*

Proof. Given $x \in U$, $v \in T_xM$ then the component of $\mathrm{d}s_x(v)$ in $T_xM \oplus T_{f(x)}N$ is equal to $\big(v, \mathrm{d}f_x(v)\big)$. Thus, s is $\mathcal{D}$-horizontal if and only if σ is ∇-parallel and $\sigma(x) = \mathrm{d}f(x)$, for all $x \in U$. The conclusion follows from Lemma 5.6. □

Lemma 5.10. *Let λ be as in the statement of Lemma 5.7 and let t_0 in the domain of λ be fixed. Assume that γ is a geodesic on M. Then λ is $\mathcal{D}$-horizontal if and only if the following conditions hold:*

- *μ is a geodesic on N;*
- *$\mu'(t_0) = \sigma(t_0)\gamma'(t_0)$;*
- *for all t, the linear map $\sigma(t) : T_{\gamma(t)}M \to T_{\mu(t)}N$ is given by the composition of ∇^M-parallel transport along γ, $\sigma(t_0)$ and ∇^N-parallel transport along μ.*

Proof. Clearly λ is $\mathcal{D}$-horizontal if and only if λ is parallel with respect to ∇ and $\mu'(t) = \sigma(t)\gamma'(t)$, for all t. The conclusion follows from Lemma 5.7 and Corollary 5.8. □

Corollary 5.11. *Let $x_0 \in M$, $y_0 \in N$ be fixed and let $\sigma_0 : T_{x_0}M \to T_{y_0}N$ be a linear map. Let $\gamma : [a, b] \to M$ be a piecewise geodesic with $\gamma(a) = x_0$. Then $\lambda : [a, b] \ni t \mapsto \big(\gamma(t), \mu(t), \sigma(t)\big) \in E$ is the horizontal lift of γ with $\lambda(a) = (x_0, y_0, \sigma_0)$ if and only if $\mu : [a, b] \to N$ is the piecewise geodesic induced by γ and σ_0 and $\sigma(t)$ is the linear map induced by $\gamma|_{[a,t]}$ and σ_0, for all t.* □

Lemma 5.12. *The curvature tensor R^E of the connection ∇ of E is given by:*

$$R^E_{(x,y)}\big((v_1, w_1), (v_2, w_2)\big)\sigma = R^N_y(w_1, w_2) \circ \sigma - \sigma \circ R^M_x(v_1, v_2),$$

for all $(x, y) \in M \times N$, $v_1, v_2 \in T_xM$, $w_1, w_2 \in T_yN$, $\sigma \in \mathrm{Lin}(T_xM, T_yN)$. □

Lemma 5.13. *Let P, Q be smooth manifolds, ∇ a connection on Q and $h : P \to Q$ be a smooth map. Given smooth vector fields X, Y in P then:*

$$\nabla_X\big(\mathrm{d}h(Y)\big) - \nabla_Y\big(\mathrm{d}h(X)\big) - \mathrm{d}h\big([X, Y]\big) = T\big(\mathrm{d}h(X), \mathrm{d}h(Y)\big),$$

where T denotes the torsion of ∇.

Proof. It is a standard computation in calculus with connections (see Proposition B.8). □

We will now compute the Levi form of the distribution $\mathcal{D}$. Given $x \in M$, $y \in N$, $\sigma \in \mathrm{Lin}(T_xM, T_yN)$, the Levi form of $\mathcal{D}$ at the point $\sigma \in E$ is a bilinear map $\mathfrak{L}^{\mathcal{D}}_\sigma : \mathcal{D}_\sigma \times \mathcal{D}_\sigma \to T_\sigma E/\mathcal{D}_\sigma$. We identify the space $\mathcal{D}_\sigma$ with T_xM by the isomorphism:

$$T_xM \ni v \longmapsto \big(v, \sigma(v), 0\big) \in \mathcal{D}_\sigma \subset T_xM \oplus T_yN \oplus \mathrm{Lin}(T_xM, T_yN) \cong T_\sigma E.$$

Moreover, the surjective linear map:

$$T_\sigma E \ni (v, w, \tau) \longmapsto (w - \sigma(v), \tau) \in T_yN \oplus \mathrm{Lin}(T_xM, T_yN) \tag{17}$$

has kernel D_σ and thus induces an isomorphism from the space $T_\sigma E/\mathcal{D}_\sigma$ onto $T_yN \oplus \mathrm{Lin}(T_xM, T_yN)$. Hence, the Levi form of $\mathcal{D}$ at σ will be identified with a bilinear map:

$$\mathfrak{L}^{\mathcal{D}}_\sigma : T_xM \times T_xM \longrightarrow T_yN \oplus \mathrm{Lin}(T_xM, T_yN).$$

We now compute $\mathfrak{L}^{\mathcal{D}}_\sigma$.

Lemma 5.14. *Given $x \in M$, $y \in N$, $\sigma \in \mathrm{Lin}(T_xM, T_yN)$, the Levi form of $\mathcal{D}$ at the point $\sigma \in E$ is given by:*

$$\begin{aligned}\mathfrak{L}^{\mathcal{D}}_\sigma(v_1, v_2) = \Big(\sigma\big(T^M(v_1, v_2)\big) - T^N\big(\sigma(v_1), \sigma(v_2)\big),&\\ \sigma \circ R^M_x(v_1, v_2) - R^N_y\big(\sigma(v_1), \sigma(v_2)\big) \circ \sigma\Big),&\end{aligned}$$

for all $v_1, v_2 \in T_xM$.

Proof. Given a smooth vector field X on M, we define a smooth vector field $\widetilde{X}$ on E by setting:

$$\widetilde{X}(x, y, \sigma) = \big(X(x), \sigma(X(x)), 0\big) \in T_xM \oplus T_yN \oplus \mathrm{Lin}(T_xM, T_yN) \cong T_\sigma E, \tag{18}$$

for all $x \in M$, $y \in N$, $\sigma \in \mathrm{Lin}(T_xM, T_yN)$. Observe that $\widetilde{X}$ is $\mathcal{D}$-horizontal.

Let $x \in M$, $y \in N$, $\sigma \in \mathrm{Lin}(T_xM, T_yN)$, $v_1, v_2 \in T_xM$ be fixed. Choose smooth vector fields X_1, X_2 on M with $X_1(x) = v_1$, $X_2(x) = v_2$. In order to compute the Levi form of $\mathcal{D}$ at the point σ it suffices to compute the Lie bracket $[\widetilde{X}_1, \widetilde{X}_2]$ at the point σ. The vector $[\widetilde{X}_1, \widetilde{X}_2]_\sigma$ is identified with an element of $T_xM \oplus T_yN \oplus \mathrm{Lin}(T_xM, T_yN)$. The component in $\mathrm{Lin}(T_xM, T_yN)$ of such vector can be computed using Lemma 4.1, since $\widetilde{X}_1$ and $\widetilde{X}_2$ are both horizontal with respect to the connection ∇ of E; thus, the component of $[\widetilde{X}_1, \widetilde{X}_2]_\sigma$ in $\mathrm{Lin}(T_xM, T_yN)$ is equal to $-R^E\big((v_1, \sigma(v_1)), (v_2, \sigma(v_2))\big)\sigma$. Let us now compute the component of $[\widetilde{X}_1, \widetilde{X}_2]_\sigma$ in $T_xM \oplus T_yN$; this is just $\mathrm{d}\pi_\sigma\big([\widetilde{X}_1, \widetilde{X}_2]_\sigma\big)$. Consider the connection $\nabla^{M\times N}$ on $M \times N$ induced from ∇^M and ∇^N; its torsion $T^{M\times N}$ is given by:

$$T^{M\times N}\big((v_1,w_1),(v_2,w_2)\big)=\big(T^M(v_1,v_2),T^N(w_1,w_2)\big).$$

We now compute $\mathrm{d}\pi_\sigma\big([\widetilde{X}_1,\widetilde{X}_2]_\sigma\big)$ using Lemma 5.13 with $P=E$, $Q=M\times N$ and $h=\pi$. We get:

$$\begin{aligned}(19)\quad \nabla^{M\times N}_{\widetilde{X}_1}\big(\mathrm{d}\pi(\widetilde{X}_2)\big)-\nabla^{M\times N}_{\widetilde{X}_2}\big(\mathrm{d}\pi(\widetilde{X}_1)\big)-\mathrm{d}\pi\big([\widetilde{X}_1,\widetilde{X}_2]\big)&\\ =\big(T^M(X_1,X_2),T^N(\sigma(X_1),\sigma(X_2))\big).&\end{aligned}$$

We compute $\nabla^{M\times N}_{\widetilde{X}_1}\big(\mathrm{d}\pi(\widetilde{X}_2)\big)$ as follows:

$$\nabla^{M\times N}_{\widetilde{X}_1}\big(\mathrm{d}\pi(\widetilde{X}_2)\big)=\tfrac{\mathrm{D}}{\mathrm{d}t}^{M\times N}\,\mathrm{d}\pi\big(\widetilde{X}_2(\lambda(t))\big),$$

where $\lambda:\,]-\varepsilon,\varepsilon[\,\to E$ is an integral curve of $\widetilde{X}_1$ with $\lambda(0)=\sigma$. Thus $\lambda(t)=\big(x(t),y(t),\sigma(t)\big)$, where $t\mapsto x(t)\in M$ is an integral curve of X_1, $y'(t)=\sigma(t)x'(t)$ and $t\mapsto\sigma(t)$ is ∇-parallel. Hence:

$$\begin{aligned}\tfrac{\mathrm{D}}{\mathrm{d}t}^{M\times N}\,\mathrm{d}\pi\big(\widetilde{X}_2(\lambda(t))\big)&=\tfrac{\mathrm{D}}{\mathrm{d}t}^{M\times N}\big(X_2(x(t)),\sigma(t)X_2(x(t))\big)\\&=\big(\tfrac{\mathrm{D}}{\mathrm{d}t}^{M}X_2(x(t)),\tfrac{\mathrm{D}}{\mathrm{d}t}^{N}[\sigma(t)X_2(x(t))]\big)\\&\overset{\sigma\text{ parallel}}{=}\big(\tfrac{\mathrm{D}}{\mathrm{d}t}^{M}X_2(x(t)),\sigma(t)\tfrac{\mathrm{D}}{\mathrm{d}t}^{M}X_2(x(t))\big).\end{aligned}$$

Evaluating at $t=0$ we obtain:

$$(20)\quad \nabla^{M\times N}_{\widetilde{X}_1}\big(\mathrm{d}\pi(\widetilde{X}_2)\big)=\tfrac{\mathrm{D}}{\mathrm{d}t}^{M\times N}\,\mathrm{d}\pi\big(\widetilde{X}_2(\lambda(t))\big)\Big|_{t=0}=\big(\nabla^M_{X_1}X_2,\sigma(\nabla^M_{X_1}X_2)\big),$$

where the righthand side of (20) is evaluated at the point x. Similarly:

$$(21)\qquad \nabla^{M\times N}_{\widetilde{X}_2}\big(\mathrm{d}\pi(\widetilde{X}_1)\big)=\big(\nabla^M_{X_2}X_1,\sigma(\nabla^M_{X_2}X_1)\big).$$

Using (19), (20) and (21) we get:

$$\begin{aligned}&\mathrm{d}\pi_\sigma\big([\widetilde{X}_1,\widetilde{X}_2]_\sigma\big)\\&=\Big([X_1,X_2]_x,\sigma([X_1,X_2]_x)+\sigma\big(T^M(X_1,X_2)\big)-T^N\big(\sigma(X_1),\sigma(X_2)\big)\Big).\end{aligned}$$

Hence, recalling Lemma 5.12:

$$\begin{aligned}(22)\ [\widetilde{X}_1,\widetilde{X}_2]_\sigma=\Big(&[X_1,X_2]_x,\\&\sigma([X_1,X_2]_x)+\sigma\big(T^M(X_1,X_2)\big)-T^N\big(\sigma(X_1),\sigma(X_2)\big),\\&\sigma\circ R^M_x(X_1,X_2)-R^N_y(\sigma(X_1),\sigma(X_2))\circ\sigma\Big).\end{aligned}$$

The conclusion follows recalling formula (17) that gives the identification between $T_\sigma E/\mathcal{D}_\sigma$ and $T_yN\oplus\mathrm{Lin}(T_xM,T_yN)$. $\square$

Corollary 5.15. *Given $x \in M$, $y \in N$, $\sigma \in \operatorname{Lin}(T_xM, T_yN)$, then the Levi form of $\mathcal{D}$ at the point $\sigma \in E$ vanishes if and only if the linear map $\sigma : T_xM \to T_yN$ relates T^M with T^N and R^M with R^N.*

Proof. It follows from Lemma 5.14. □

Proof of Theorems 5.1 and 5.3. It follows from Theorems 2.5 and 3.11, keeping in mind Examples 2.2 and 3.1, Lemmas 5.9 and 5.10, and Corollary 5.15. □

5.2. **Higher order Cartan–Ambrose–Hicks theorem.** Given a tensor field τ on a manifold endowed with a connection ∇, we denote by $\nabla^{(r)}\tau$ its r-th covariant derivative, for $r \geq 1$; we set $\nabla^{(0)}\tau = \tau$.

Theorem 5.16. *Let M, N be real-analytic manifolds endowed with real-analytic connections ∇^M and ∇^N, respectively. Let $x_0 \in M$, $y_0 \in N$ be given and let $\sigma_0 : T_{x_0}M \to T_{y_0}N$ be a linear map. If for all $r \geq 0$ the linear map σ_0 relates $\nabla^{(r)}T^M_{x_0}$ with $\nabla^{(r)}T^N_{y_0}$ and $\nabla^{(r)}R^M_{x_0}$ with $\nabla^{(r)}R^N_{y_0}$ then there exists a real-analytic affine map $f : U \to N$ defined on an open neighborhood U of x_0 in M satisfying $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.*

Proof. We will apply Theorem 2.7 to the distribution $\mathcal{D}$ on E defined in (16). As before, for $x \in M$, $y \in N$, $\sigma \in \operatorname{Lin}(T_xM, T_yN)$, we use the identification:

$$T_\sigma E \cong T_xM \oplus T_yN \oplus \operatorname{Lin}(T_xM, T_yN).$$

Given a smooth vector field X on M, we define a $\mathcal{D}$-horizontal vector field $\widetilde{X}$ on E as in (18). Recall that for $X_1, X_2 \in \Gamma(TM)$, the bracket $[\widetilde{X}_1, \widetilde{X}_2]$ was computed in the proof of Lemma 5.14 (see (22)). By Remark 2.8, the thesis will follow once we show that the iterated brackets:

$$[\widetilde{X}_{r+1}, \ldots, \widetilde{X}_1] \stackrel{\text{def}}{=} [\widetilde{X}_{r+1}, [\widetilde{X}_r, \ldots, [\widetilde{X}_2, \widetilde{X}_1] \cdots]], \tag{23}$$

evaluated at $\sigma_0 \in E$ are in $\mathcal{D}_{\sigma_0}$, for all $X_1, \ldots, X_r \in \Gamma(TM)$ and all $r \geq 1$. For $r \geq 0$, $x \in M$, $y \in N$, $\sigma \in \operatorname{Lin}(T_xM, T_yN)$ we set:

$$\begin{aligned}\mathfrak{T}^{(r)}_\sigma(X_1, \ldots, X_{r+2}) &= \sigma\big(\nabla^{(r)}T^M(X_1, \ldots, X_{r+2})\big)\\ &\quad - \nabla^{(r)}T^N\big(\sigma(X_1), \ldots, \sigma(X_{r+2})\big) \in T_yN,\end{aligned}$$

$$\begin{aligned}\mathfrak{R}^{(r)}_\sigma(X_1, \ldots, X_{r+2}) &= \sigma \circ \nabla^{(r)}R^M(X_1, \ldots, X_{r+2})\\ &\quad - \nabla^{(r)}R^N\big(\sigma(X_1), \ldots, \sigma(X_{r+2})\big) \circ \sigma \in \operatorname{Lin}(T_xM, T_yN),\end{aligned}$$

for all $X_1, \ldots, X_{r+2} \in T_xM$. The hypotheses of the theorem say that $\mathfrak{T}^{(r)}_{\sigma_0}$ and $\mathfrak{R}^{(r)}_{\sigma_0}$ vanish for all $r \geq 0$. Observe that $\mathfrak{T}^{(r)}$ and $\mathfrak{R}^{(r)}$ are, respectively, sections along the map $\pi : E \to M \times N$ of the vector bundles over $M \times N$ given by:

$$\Big(\bigotimes_{r+2}(\pi_1^*TM)^*\Big) \otimes \pi_2^*TN \quad \text{and} \quad \Big(\bigotimes_{r+2}(\pi_1^*TM)^*\Big) \otimes (\pi_1^*TM)^* \otimes \pi_2^*TN,$$

where π_1 and π_2 denote the projections of the product $M \times N$.

Our plan is to show that the iterated bracket (23) can be written in the form:

$$\begin{aligned}(24)\quad [\widetilde{X}_{r+1}, \ldots, \widetilde{X}_1] &= \big(0, \mathfrak{T}^{(r-1)}(X_{r+1}, \ldots, X_1), \mathfrak{R}^{(r-1)}(X_{r+1}, \ldots, X_1)\big) \\ &\quad + \big(0, \mathcal{L}^{(r)}(\mathfrak{T}^{(0)}, \mathfrak{R}^{(0)}, \ldots, \mathfrak{T}^{(r-2)}, \mathfrak{R}^{(r-2)})\big) + \text{terms in } \Gamma^{(r-1)}(\mathcal{D}),\end{aligned}$$

for all $r \geq 1$, where $\mathcal{L}^{(r)}$ is a section along the map $\pi : E \to M \times N$ of the vector bundle over $M \times N$ given by:

$$\begin{aligned}\mathrm{Lin}\Big(\bigoplus_{i=0}^{r-2}\Big[\Big(\bigotimes_{i+2}(\pi_1^*TM)^*\Big)\otimes\pi_2^*TN\oplus\Big(\bigotimes_{i+2}(\pi_1^*TM)^*\Big)\otimes(\pi_1^*TM)^*\otimes\pi_2^*TN\Big],& \\ \pi_2^*TN \oplus (\pi_1^*TM)^* \otimes \pi_2^*TN\Big).&\end{aligned}$$

Once formula (24) is proven, the conclusion follows easily by induction on r. We will now conclude the proof by showing formula (24) by induction on r. For $r = 1$, we have (recall (22)):

$$\begin{aligned}[\widetilde{X}_2, \widetilde{X}_1] &= \Big(0, \sigma\big(T^M(X_2, X_1)\big) - T^N\big(\sigma(X_2), \sigma(X_1)\big), \\ &\qquad \sigma \circ R^M(X_2, X_1) - R^N(\sigma(X_2), \sigma(X_1)) \circ \sigma\Big) + \text{terms in } \Gamma^0(\mathcal{D}) \\ &= \big(0, \mathfrak{T}^{(0)}(X_2, X_1), \mathfrak{R}^{(0)}(X_2, X_1)\big) + \text{terms in } \Gamma^0(\mathcal{D}),\end{aligned}$$

proving the base of the induction. The induction step can be proven by applying $\mathrm{ad}_{\widetilde{X}_{r+2}}$ to both sides of (24), keeping in mind Lemma 5.17 below and the following formulas:

$$\begin{aligned}(\nabla_{\mathrm{hor}}\mathfrak{T}^{(i)})_\sigma\big(Z, \sigma(Z)\big) &= \mathfrak{T}^{(i+1)}(Z, \cdots), \\ (\nabla_{\mathrm{hor}}\mathfrak{R}^{(i)})_\sigma\big(Z, \sigma(Z)\big) &= \mathfrak{R}^{(i+1)}(Z, \cdots),\end{aligned}$$

where Z is a vector field on M. This concludes the proof. □

Lemma 5.17. *Let A be a section along the map $\pi_2 \circ \pi : E \to N$ of the tangent bundle of N and let B be a section along the map $\pi : E \to M \times N$ of the vector bundle E, so that $(0, A, B)$ is a vector field on the manifold*

E. Let Z be a vector field on M. Then:

$$[\widetilde{Z},(0,A,B)]_\sigma = \Big(0, \nabla^N_{\mathrm{hor}}A\big(Z,\sigma(Z)\big) - B(Z) - T^N\big(\sigma(Z),A\big),\\ \nabla_{\mathrm{hor}}B\big(Z,\sigma(Z)\big) - R^N\big(\sigma(Z),A\big)\circ\sigma\Big), \quad \sigma\in E,$$

where $\nabla^N_{\mathrm{hor}}A$ (resp., $\nabla_{\mathrm{hor}}B$) denotes the restriction of $\nabla^N A$ (resp., of ∇B) to the horizontal subbundle of TE determined by the connection of E.

Proof. We compute the horizontal component of $[\widetilde{Z},(0,A,B)]$ using Lemma 5.13 with $P=E$, $Q=M\times N$ and $h=\pi$. We have:

$$\mathrm{d}\pi_\sigma[(Z,\sigma(Z),0),(0,A,B)] = \nabla^{M\times N}_{(Z,\sigma(Z),0)}(0,A) - \nabla^{M\times N}_{(0,A,B)}\big(Z,\sigma(Z)\big)\\ - T^{M\times N}\big((Z,\sigma(Z)),(0,A)\big).$$

Clearly:

$$T^{M\times N}\big((Z,\sigma(Z)),(0,A)\big) = \big(T^M(Z,0),T^N(\sigma(Z),A)\big) = \big(0,T^N(\sigma(Z),A)\big)$$

and:

$$\nabla^{M\times N}_{(Z,\sigma(Z),0)}(0,A) = \big(0,\nabla^N_{\mathrm{hor}}A(Z,\sigma(Z))\big).$$

Let $t\mapsto\big(x(t),y(t),\sigma(t)\big)$ be an integral curve of $(0,A,B)$, i.e., $t\mapsto x(t)$ is constant, $y'=A$ and $\frac{\mathrm{D}}{\mathrm{d}t}\sigma=B$. We compute:

$$\nabla^{M\times N}_{(0,A,B)}\big(Z,\sigma(Z)\big) = \tfrac{\mathrm{D}}{\mathrm{d}t}^{M\times N}\big(Z_{x(t)},\sigma(t)Z_{x(t)}\big) = \big(0,B(Z)\big).$$

Let us now compute the vertical component of $[\widetilde{Z},(0,A,0)]$. Since both $\widetilde{Z}$ and $(0,A,0)$ are in the horizontal subbundle of TE determined by the connection of E, the vertical component of $[\widetilde{Z},(0,A,0)]$ can be directly computed using Lemmas 4.1 and 5.12, as follows:

$$\text{vertical component of } [\widetilde{Z},(0,A,0)]_\sigma = -R^E\big((Z,\sigma(Z)),(0,A)\big)\sigma\\ = \sigma\circ R^M(Z,0) - R^N\big(\sigma(Z),A\big)\circ\sigma = -R^N\big(\sigma(Z),A\big)\circ\sigma.$$

Finally, we compute the vertical component of $[\widetilde{Z},(0,0,B)]$. Let W be a vector field on M and α be a 1-form on N; we define a map $f_{W,\alpha}:E\to\mathbb{R}$ by setting:

$$f_{W,\alpha}(\sigma) = \alpha\big(\sigma(W)\big).$$

Let $x\in M$, $y\in N$, $\sigma\in\mathrm{Lin}(T_xM,T_yN)$ be fixed and assume that $\nabla^M W(x)=0$, $\nabla^N\alpha(y)=0$, so that $\mathrm{d}f_{W,\alpha}(\sigma)$ annihilates the horizontal subspace of $T_\sigma E$ determined by the connection of E. We compute:

$$
\begin{aligned}
(0,0,B)(f_{W,\alpha}) &= \alpha\big(B(W)\big),\\
\widetilde{Z}(f_{W,\alpha}) &= (\nabla_{\sigma(Z)}\alpha)\big(\sigma(W)\big) + \alpha\big(\sigma(\nabla_Z W)\big),\\
\widetilde{Z}\big((0,0,B)(f_{W,\alpha})\big)_\sigma &= (\nabla_{\sigma(Z)}\alpha)_x\big(B(W)\big) + \alpha\big(\nabla_{\mathrm{hor}} B_\sigma(Z,\sigma(Z))(W)\big)\\
&\quad + \alpha\big(B(\nabla_{Z_x} W)\big) = \alpha\big(\nabla_{\mathrm{hor}} B_\sigma(Z,\sigma(Z))(W)\big),\\
(0,0,B)\big(\widetilde{Z}(f_{W,\alpha})\big)_\sigma &= (\nabla_{B(Z)}\alpha)_x\big(\sigma(W)\big) + (\nabla_{\sigma(Z)}\alpha)_x\big(B(W)\big)\\
&\quad + \alpha\big(B(\nabla_{Z_x} W)\big) = 0,
\end{aligned}
$$

so that:

$$
[\widetilde{Z},(0,0,B)](f_{W,\alpha}) = \alpha\big(\nabla_{\mathrm{hor}} B_\sigma(Z,\sigma(Z))(W)\big).
$$

Hence, the vertical component of $[\widetilde{Z},(0,0,B)]_\sigma$ is equal to $\nabla_{\mathrm{hor}} B_\sigma\big(Z,\sigma(Z)\big)$. This concludes the proof. □

Proposition 5.18. *Let M, N be real-analytic manifolds endowed with real-analytic connections ∇^M and ∇^N, respectively. Assume that ∇^N is geodesically complete and that M is (connected and) simply-connected. Then every affine map $f : U \to N$ defined on a nonempty connected open subset U of M extends to an affine map from M to N. In particular, if in addition $x_0 \in M$, $y_0 \in N$, $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$ satisfy the hypotheses of Theorem 5.16 then there exists an affine map $f : M \to N$ with $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.*

Proof. If $\mathcal{D}$ is the distribution on E defined in (16) then, by Lemma 5.9, $s(x) = \big(f(x), \mathrm{d}f(x)\big)$ is a $\mathcal{D}$-horizontal section of $\pi : E \to M$ defined in U. The geodesical completeness of ∇^N guarantees that hypothesis (b) of Proposition 3.12 is satisfied; hence, such proposition gives a global horizontal section of π. □

A special case of Proposition 5.18, namely when the manifolds M and N have the same dimension and σ_0 is an isomorphism, is proved in [2, p. 259–261].

An *affine symmetry* around a point $x_0 \in M$ is an affine map $f : U \to M$ defined in an open neighborhood U of x_0 with $f(x_0) = x_0$ and $\mathrm{d}f(x_0) = -\mathrm{Id}$.

Corollary 5.19. *Let M be a real-analytic manifold endowed with a real-analytic connection ∇. Let $x_0 \in M$ be fixed. Then there exists an affine symmetry around x_0 if and only if:*

$$
\nabla^{(2r)} T_{x_0} = 0, \quad \textit{and} \quad \nabla^{(2r+1)} R_{x_0} = 0, \quad \textit{for all } r \geq 0. \tag{25}
$$

Moreover, if M is (connected and) simply-connected and complete then condition (25) is equivalent to the existence of a globally defined affine symmetry $f : M \to M$ around x_0.

Proof. Apply Theorem 5.16 with $M = N$, $y_0 = x_0$ and $\sigma_0 = -\mathrm{Id}$. For the global result apply Proposition 5.18. $\square$

Appendix A. A globalization principle

Definition A.1. Let X, $\widetilde{X}$ be topological spaces and $\pi : \widetilde{X} \to X$ be a map. An open subset $U \subset X$ is called a *fundamental open subset* of X if $\pi^{-1}(U)$ equals a disjoint union $\bigcup_{i\in I} U_i$ of open subsets U_i of $\widetilde{X}$ such that $\pi|_{U_i} : U_i \to U$ is a homeomorphism for all $i \in I$. We say that π is a *covering map* if X can be covered by fundamental open subsets.

Obviously every covering map is a local homeomorphism.

Given a local homeomorphism $\pi : \widetilde{X} \to X$ then by a *local section* of π we mean a continuous map $s : U \to \widetilde{X}$ defined on an open subset of X with $\pi \circ s = \mathrm{Id}_U$.

Lemma A.2. *Let X, $\widetilde{X}$ be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism. Assume that $\widetilde{X}$ is Hausdorff. Let U be a connected open subset of X satisfying the following property:*

(*) *for every $x \in U$ and every $\tilde{x} \in \widetilde{X}$ with $\pi(\tilde{x}) = x$ there exists a local section $s : U \to \widetilde{X}$ of π with $s(x) = \tilde{x}$.*

Then U is a fundamental open subset of X.

Proof. Let $\mathcal{S}$ be the set of all local sections of π defined in U. We claim that:

$$\pi^{-1}(U) = \bigcup_{s\in\mathcal{S}} s(U).$$

Indeed, if $s \in \mathcal{S}$ then obviously $s(U) \subset \pi^{-1}(U)$; moreover, given $\tilde{x} \in \pi^{-1}(U)$ then $x = \pi(\tilde{x}) \in U$ and by property $(*)$ there exists $s \in \mathcal{S}$ with $s(x) = \tilde{x}$. Thus $\tilde{x} \in s(U)$. This proves the claim. Now observe that $s(U)$ is open in $\widetilde{X}$ for all $s \in \mathcal{S}$; moreover, $\pi|_{s(U)} : s(U) \to U$ is a homeomorphism, being the inverse of $s : U \to s(U)$. To complete the proof, we show that the union $\bigcup_{s\in\mathcal{S}} s(U)$ is disjoint. Pick $s, s' \in \mathcal{S}$ with $s(U) \cap s'(U) \neq \emptyset$. Then there exists $x, y \in U$ with $s(x) = s'(y)$. Observe that:

$$x = \pi\big(s(x)\big) = \pi\big(s'(y)\big) = y,$$

and thus $s(x) = s'(x)$. Since U is connected and $\widetilde{X}$ is Hausdorff it follows that $s = s'$. $\square$

Corollary A.3. *Let X, $\widetilde{X}$ be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism. Assume that $\widetilde{X}$ is Hausdorff and that X is locally connected. If every point of X has an open neighborhood satisfying property $(*)$ then π is a covering map.* $\square$

Let X be a topological space. A *pre-sheaf* on X is a map $\mathfrak{P}$ that assigns to each open subset $U \subset X$ a set $\mathfrak{P}(U)$ and to each pair of open subsets $U, V \subset X$ with $V \subset U$ a map $\mathfrak{P}_{U,V} : \mathfrak{P}(U) \to \mathfrak{P}(V)$ such that the following properties hold:

- for every open subset $U \subset X$ the map $\mathfrak{P}_{U,U}$ is the identity map of the set $\mathfrak{P}(U)$;
- given open sets, $U, V, W \subset X$ with $W \subset V \subset U$ then:

$$\mathfrak{P}_{V,W} \circ \mathfrak{P}_{U,V} = \mathfrak{P}_{U,W}.$$

We say that the pre-sheaf $\mathfrak{P}$ is *nontrivial* if there exists a nonempty open subset U of X with $\mathfrak{P}(U) \neq \emptyset$.

A.1. **Example.** For each open subset U of X let $\mathfrak{P}(U)$ be the set of all continuous maps $f : U \to \mathbb{R}$. Given open subsets U, V of X with $V \subset U$ we set $\mathfrak{P}_{U,V}(f) = f|_V$, for all $f \in \mathfrak{P}(U)$. Then $\mathfrak{P}$ is a pre-sheaf over X.

A *sheaf* over a topological space X is a pair $(\mathcal{S}, \pi)$, where $\mathcal{S}$ is a topological space and $\pi : \mathcal{S} \to X$ is a local homeomorphism.

Let $\mathfrak{P}$ be a pre-sheaf over a topological space X. Given a point $x \in X$, consider the disjoint union of all sets $\mathfrak{P}(U)$, where U is an open neighborhood of x in X. We define an equivalence relation $\sim$ on such disjoint union as follows; given $f_1 \in \mathfrak{P}(U_1)$, $f_2 \in \mathfrak{P}(U_2)$, where U_1, U_2 are open neighborhoods of x in X then $f_1 \sim f_2$ if and only if there exists an open neighborhood V of x contained in $U_1 \cap U_2$ such that $\mathfrak{P}_{U_1,V}(f_1) = \mathfrak{P}_{U_2,V}(f_2)$. If U is an open neighborhood of x in X and $f \in \mathfrak{P}(U)$ then the equivalence class of f corresponding to the equivalence relation $\sim$ will be denote by $[f]_x$ and will be called the *germ* of f at the point x. We set:

$$\mathfrak{S}_x = \big\{[f]_x : f \in \mathfrak{P}(U), \text{ for some open neighborhood } U \text{ of } x \text{ in } X\big\}.$$

Let $\mathfrak{S}$ denote the disjoint union of all $\mathfrak{S}_x$, with $x \in X$. Let $\pi : \mathfrak{S} \to X$ denote the map that carries $\mathfrak{S}_x$ to the point x. Our goal now is to define a topology on $\mathfrak{S}$. Given an open subset $U \subset X$ and an element $f \in \mathfrak{P}(U)$ we set:

$$\mathcal{V}(f) = \big\{[f]_x : x \in U\big\} \subset \mathfrak{S}.$$

The set:

$$\big\{\mathcal{V}(f) : f \in \mathfrak{P}(U),\ U \text{ an open subset of } X\big\}$$

is a basis for a topology on $\mathfrak{S}$; moreover, if $\mathfrak{S}$ is endowed with such topology, the map $\pi : \mathfrak{S} \to X$ is a local homeomorphism, so that $(\mathfrak{S}, \pi)$ is a sheaf over X. We call $(\mathfrak{S}, \pi)$ the *sheaf of germs* corresponding to the pre-sheaf $\mathfrak{P}$. Observe that if U is an open subset of X and $f \in \mathfrak{P}(U)$ then the map $\hat{f} : U \to \mathfrak{S}$ defined by

$$\hat{f}(x) = [f]_x, \quad x \in U,$$

is a local section of the sheaf of germs defined in U.

Definition A.4. We say that the pre-sheaf $\mathfrak{P}$ has the *localization property* if, given a family $(U_i)_{i \in I}$ of open subsets of X and setting $U = \bigcup_{i \in I} U_i$ then the map:

$$\mathfrak{P}(U) \ni f \longmapsto \big(\mathfrak{P}_{U,U_i}(f)\big)_{i \in I} \in \prod_{i \in I} \mathfrak{P}(U_i) \tag{26}$$

is injective and its image consists of all the families $(f_i)_{i \in I}$ in $\prod_{i \in I} \mathfrak{P}(U_i)$ such that $\mathfrak{P}_{U_i, U_i \cap U_j}(f_i) = \mathfrak{P}_{U_j, U_i \cap U_j}(f_j)$, for all $i, j \in I$.

Remark A.5. If the pre-sheaf $\mathfrak{P}$ has the localization property then for every local section $s : U \to \mathfrak{S}$ of its sheaf of germs $\mathfrak{S}$ there exists a unique $f \in \mathfrak{P}(U)$ such that $\hat{f} = s$.

Definition A.6. We say that the pre-sheaf $\mathfrak{P}$ has the *uniqueness property* if for every connected open subset $U \subset X$ and every nonempty open subset $V \subset U$ the map $\mathfrak{P}_{U,V}$ is injective. We say that an open subset $U \subset X$ has the *extension property with respect to the pre-sheaf* $\mathfrak{P}$ if for every connected nonempty open subset V of U the map $\mathfrak{P}_{U,V}$ is surjective. We say that the pre-sheaf $\mathfrak{P}$ has the *extension property* if X can be covered by open sets having the extension property with respect to $\mathfrak{P}$.

Remark A.7. If the pre-sheaf $\mathfrak{P}$ has the uniqueness property and if X is locally connected and Hausdorff then the space $\mathfrak{S}$ is Hausdorff. If X is locally connected and if U is an open subset of X having the extension property with respect to the pre-sheaf $\mathfrak{P}$ then U has the property $(*)$ with respect to the local homeomorphism $\pi : \mathfrak{S} \to X$. It follows from Lemma A.2 that if X is Hausdorff and locally connected and if the pre-sheaf $\mathfrak{P}$ has the uniqueness property and the extension property then the map $\pi : \mathfrak{S} \to X$ is a covering map.

Proposition A.8. *Assume that X is Hausdorff, locally arc-connected, connected, and simply-connected. If $\mathfrak{P}$ is a pre-sheaf over X satisfying the localization property, the uniqueness property and the extension property then the open set X has the extension property for $\mathfrak{P}$, i.e., for every nonempty open connected subset V of X the map $\mathfrak{P}_{X,V} : \mathfrak{P}(X) \to \mathfrak{P}(V)$ is surjective. In particular, if $\mathfrak{P}$ is nontrivial then the set $\mathfrak{P}(X)$ is nonempty.*

Proof. Let V be a nonempty open connected subset of X and let $f \in \mathfrak{P}(V)$ be fixed. We will show that f is in the image of $\mathfrak{P}_{X,V}$. Let $\pi : \mathfrak{S} \to X$ denote the sheaf of germs of $\mathfrak{P}$. By Remark A.7 π is a covering map. Choose an arbitrary point $x_0 \in V$ and let $\mathfrak{S}_0$ be the arc-connected component of $[f]_{x_0}$ in $\mathfrak{S}$. Since X is locally arc-connected, the restriction of π to $\mathfrak{S}_0$ is again a covering map. By the connectedness and simply-connectedness of X, $\pi|_{\mathfrak{S}_0} : \mathfrak{S}_0 \to X$ is a homeomorphism. The inverse of $\pi|_{\mathfrak{S}_0}$ is therefore a global section $s : X \to \mathfrak{S}$ and, by Remark A.5, there exists $g \in \mathfrak{P}(X)$ with $\hat{g} = s$. Now $[g]_{x_0} = \hat{g}(x_0) = s(x_0) = [f]_{x_0}$ and hence, by the uniqueness property, $\mathfrak{P}_{X,V}(g) = f$. □

Appendix B. A crash course on calculus with connections

Given a smooth vector bundle $\pi : E \to M$ over a smooth manifold M, we will denote by $\Gamma(E)$ the space of all smooth sections $s : M \to E$ of E. Observe that $\Gamma(E)$ is a real vector space and it is a module over the commutative ring $C^\infty(M)$ of all smooth maps $f : M \to \mathbb{R}$. Given an open subset U of M, we denote by $E|_U$ the restriction of the vector bundle E to U, i.e., $E|_U = \pi^{-1}(U)$.

Definition B.1. A *connection* on a vector bundle $\pi : E \to M$ is a $\mathbb{R}$-bilinear map:

$$\nabla : \Gamma(TM) \times \Gamma(E) \ni (X, s) \longmapsto \nabla_X s \in \Gamma(E)$$

that is $C^\infty(M)$-linear in the variable X and satisfies the Leibnitz derivative rule:

$$\nabla_X(fs) = X(f)s + f\nabla_X s,$$

for all $X \in \Gamma(TM)$, $s \in \Gamma(E)$, $f \in C^\infty(M)$.

B.1. **Example.** If E_0 is a fixed real finite-dimensional vector space and $E = M \times E_0$ is a trivial vector bundle over M then a section s of E can be identified with a map $s : M \to E_0$ and a connection on E can be defined by:

$$\nabla_X s = \mathrm{d}s(X), \tag{27}$$

for all $X \in \Gamma(TM)$. We call (27) the *standard connection* of the trivial bundle E.

It follows from the $C^\infty(TM)$-linearity of ∇ in the variable X that $\nabla_X s(x)$ depends only of the value of X at the point $x \in M$, i.e., if $X(x) = X'(x)$ then $\nabla_X s(x) = \nabla_{X'} s(x)$. Given $s \in \Gamma(E)$, $x \in M$ and $v \in T_xM$, we set:

$$\nabla_v s = \nabla_X s(x),$$

where $X \in \Gamma(TM)$ is an arbitrary vector field with $X(x) = v$. For all $x \in M$ we denote by $\nabla s(x) : T_xM \to E_x$ the linear map given by $v \mapsto \nabla_v s$. Thus, given $s \in \Gamma(E)$, we obtain a smooth section ∇s of $TM^* \otimes E$.

It follows from the Leibnitz rule that if $U \subset M$ is an open subset then the restriction of $\nabla_X s$ to U depends only of the restriction of s to U. Thus, given an open subset U of M, a connection ∇ on E induces a unique connection ∇^U on $E|_U$ such that:

$$\nabla^U_v(s|_U) = \nabla_v s, \tag{28}$$

for all $s \in \Gamma(E)$, $v \in TU$.

Remark B.2. Given connections ∇ and ∇' on a vector bundle $\pi : E \to M$ then their difference is a tensor; more explicitly:

$$\mathfrak{t}(X, s) = \nabla_X s - \nabla'_X s \in \Gamma(E), \quad X \in \Gamma(TM),\ s \in \Gamma(E),$$

is $C^\infty(M)$-bilinear and hence defines a smooth section $\mathfrak{t}$ of the vector bundle $TM^* \otimes E^* \otimes E$. Moreover, if ∇ is a connection on E and $\mathfrak{t}$ is a smooth section of $TM^* \otimes E^* \otimes E$ then $\nabla + \mathfrak{t}$ is also a connection on E. If $\mathfrak{t}$ is a section of $TM^* \otimes E^* \otimes E$ then, given $x \in M$, $v \in T_xM$, we identify $\mathfrak{t}(v)$ with a linear operator on the fiber E_x.

Given vector bundles $\pi : E \to M$, $\tilde{\pi} : \widetilde{E} \to M$ over the same base manifold M then a *vector bundle morphism* is a smooth map $L : E \to \widetilde{E}$ such that $\tilde{\pi} \circ L = \pi$ and such that $L|_{E_x} : E_x \to \widetilde{E}_x$ is a linear map, for all $x \in M$. We will denote the restriction of L to E_x by L_x. If $L : E \to \widetilde{E}$ is a vector bundle morphism such that L_x is an isomorphism for all $x \in M$ then we call L a *vector bundle isomorphism*. If A is an open subset of E and $L : A \to \widetilde{E}$ is a smooth map such that $\tilde{\pi} \circ L = \pi|_A$ then we call L a *fiber bundle morphism*. Given $x \in M$, we write $A_x = A \cap E_x$ and $L_x = L|_{A_x} : A_x \to \widetilde{E}_x$.

Definition B.3. Given vector bundles $\pi : E \to M$, $\tilde{\pi} : \widetilde{E} \to M$ over the same base manifold M, a vector bundle morphism $L : E \to \widetilde{E}$ and connections ∇, $\widetilde{\nabla}$ on E and $\widetilde{E}$ respectively then we say that ∇ and $\widetilde{\nabla}$ are *L-related* if:

$$\widetilde{\nabla}_X\big(L(s)\big) = L(\nabla_X s),$$

for all $X \in \Gamma(TM)$, $s \in \Gamma(E)$.

In what follows, we will deal with several constructions involving connections on different vector bundles. In order to avoid heavy notations, we will usually denote all these connections by the symbol ∇; it should be clear from the context which connection the symbol ∇ refers to. For instance, formula (28) will be rewritten in the following simpler form:

$$\nabla_v(s|_U) = \nabla_v s.$$

Definition B.4. Given a connection ∇ on a vector bundle $\pi : E \to M$, then the *curvature tensor* of ∇ is defined by:

$$R(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s \in \Gamma(E), \tag{29}$$

for all $X, Y \in \Gamma(TM)$, $s \in \Gamma(E)$.

Since the righthand side of (29) is $C^\infty(M)$-linear in the variables X, Y and s, it follows that R can be identified with a smooth section of the vector bundle $TM^* \otimes TM^* \otimes E^* \otimes E$. Clearly, $R(X,Y)s$ is anti-symmetric in the variables X and Y.

The notion of torsion is usually defined only for connection on tangent bundles. We will present a slight generalization of this notion.

Definition B.5. Let $\pi : E \to M$ be a smooth vector bundle and let $\iota : TM \to E$ be a vector bundle morphism. Given a connection ∇ on E then the *ι-torsion* of ∇ is defined by:

$$T^\iota(X,Y) = \nabla_X\big(\iota(Y)\big) - \nabla_Y\big(\iota(X)\big) - \iota\big([X,Y]\big) \in \Gamma(E), \tag{30}$$

for all $X, Y \in \Gamma(TM)$. If $E = TM$ and ι is the identity map of TM, we will write simply T and call it the *torsion* of ∇.

Again, the righthand side of (30) is $C^\infty(M)$-linear on the variables X and Y, so that T^ι can be identified with a smooth section of the vector bundle $TM^* \otimes TM^* \otimes E$. Clearly, $T^\iota(X,Y)$ is anti-symmetric in X and Y.

In what follows we will study some natural constructions with vector bundles endowed with connections and we will present some formulas for the computation of torsions and curvatures. We will consider constructions that act on the basis of the vector bundles and constructions that act on their fibers.

Given smooth manifolds, M, N, a smooth vector bundle $\pi : E \to M$ over M and a smooth map $f : N \to M$ then we denote by f^*E the *pull-back of E by f* which is a vector bundle over N whose fiber at a point $x \in N$ is equal to $E_{f(x)}$. Observe that there is a natural identification of smooth sections of the bundle f^*E with smooth *sections of E along f*, i.e., smooth maps $s : N \to E$ such that $\pi \circ s = f$. Notice that every smooth section $s : M \to E$ of E gives rise to a smooth section of E along f given by $s \circ f : N \to E$; we may thus identify $s \circ f$ with a section of f^*E.

Proposition B.6. *Given smooth manifolds M, N, a smooth vector bundle E over M endowed with a connection ∇ and a smooth map $f : N \to M$*

then there exists a unique connection $f^*\nabla$ on the pull-back bundle f^*E such that:

$$(f^*\nabla)_v(s\circ f) = \nabla_{\mathrm{d}f(v)}s, \tag{31}$$

for all $s \in \Gamma(E)$ and all $v \in TN$.

The next result follows easily from Proposition B.6.

Proposition B.7. *Let P, N, M be smooth manifolds, E be a vector bundle over M endowed with a connection ∇ and $g : P \to N$, $f : N \to M$ be smooth maps. Then:*

$$(f\circ g)^*\nabla = f^*(g^*\nabla); \tag{32}$$

moreover, if $i : U \to M$ denotes the inclusion map of an open subset U of M then i^*E can be naturally identified with the bundle $E|_U$ and $i^*\nabla$ coincides with the induced connection ∇^U.

Identity (32) can be interpreted as a *chain rule* as follows; given a section $s : N \to E$ of E along f and $v \in TP$ then:

$$\big((f\circ g)^*\nabla\big)_v(s\circ g) \overset{\text{by (32)}}{=} \big(g^*(f^*\nabla)\big)_v(s\circ g) \overset{\text{by (31)}}{=} (f^*\nabla)_{\mathrm{d}g(v)}s.$$

We have the following natural formula to compute the curvature and the torsion of a pull-back connection.

Proposition B.8. *Given smooth manifolds M, N, a smooth vector bundle E over M endowed with a connection ∇ and a smooth map $f : N \to M$ then the curvature tensor of $f^*\nabla$ is given by:*

$$R^{f^*\nabla}_x(v,w)e = R^{\nabla}_{f(x)}\big(\mathrm{d}f(x)v, \mathrm{d}f(x)w\big)e,$$

for all $x \in N$, $v, w \in T_xN$, $e \in (f^*E)_x = E_{f(x)}$. Moreover, given a smooth vector bundle morphism $\iota : TM \to E$, then $\iota \circ \mathrm{d}f : TN \to E$ is identified with a vector bundle morphism $\tilde{\iota} : TN \to f^*E$ and the the following formula holds:

$$T^{\tilde{\iota}}_x(v,w) = T^{\iota}_{f(x)}\big(\mathrm{d}f(x)v, \mathrm{d}f(x)w\big), \tag{33}$$

for all $x \in N$, $v, w \in T_xN$.

Observe that if $E = TM$ and ι is the identity of TM then formula (31) means that:

$$(f^*\nabla)_X\big(\mathrm{d}f(Y)\big) - (f^*\nabla)_Y\big(\mathrm{d}f(X)\big) - \mathrm{d}f\big([X,Y]\big) = T\big(\mathrm{d}f(X), \mathrm{d}f(Y)\big),$$

for all $X, Y \in \Gamma(TN)$.

Now we consider constructions acting on the fibers of the vector bundles. To this aim, we need some categorical language. Given an integer $n \geq 1$, we denote by $\underline{\mathfrak{Vec}}^n$ the category whose objects are n-tuples $(V_i)_{i=1}^n$ of real finite-dimensional vector spaces and whose morphisms from $(V_i)_{i=1}^n$ to $(W_i)_{i=1}^n$ are n-tuples $(T_i)_{i=1}^n$ of *vector space isomorphisms* $T_i : V_i \to W_i$. We set $\underline{\mathfrak{Vec}}^1 = \underline{\mathfrak{Vec}}$. A functor $\mathfrak{F} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ is called *smooth* if for any object $(V_i)_{i=1}^n$ of $\underline{\mathfrak{Vec}}^n$ the map:

$$\mathfrak{F} : \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n) \longrightarrow \mathrm{GL}\big(\mathfrak{F}(V_1, \ldots, V_n)\big) \tag{34}$$

is smooth. Observe that (34) is a Lie group homomorphism; its differential at the identity is a Lie algebra homomorphism that will be denoted by:

$$\overline{\mathfrak{F}} : \mathrm{gl}(V_1) \times \cdots \times \mathrm{gl}(V_n) \longrightarrow \mathrm{gl}\big(\mathfrak{F}(V_1, \ldots, V_n)\big).$$

Given vector bundles $E^1, \ldots, E^n$ over a smooth manifold M we obtain naturally a new vector bundle $\mathfrak{F}(E^1, \ldots, E^n)$ over M whose fiber at a point $x \in M$ is equal to $\mathfrak{F}(E^1_x, \ldots, E^n_x)$. Given a smooth manifold N and a smooth map $f : N \to M$, we may identify vector bundles $f^*\big(\mathfrak{F}(E^1, \ldots, E^n)\big)$ and $\mathfrak{F}(f^*E^1, \ldots, f^*E^n)$. Given vector bundle isomorphisms $L^i : E^i \to \widetilde{E}^i$, $i = 1, \ldots, n$, then we obtain a vector bundle isomorphism $L = \mathfrak{F}(T^1, \ldots, T^n)$ from $\mathfrak{F}(E^1, \ldots, E^n)$ to $\mathfrak{F}(\widetilde{E}^1, \ldots, \widetilde{E}^n)$ by setting:

$$L_x = \mathfrak{F}(L^1_x, \ldots, L^n_x),$$

for all $x \in M$.

We have the following functorial construction for connections.

Proposition B.9. *Given an integer $n \geq 1$ and a smooth functor $\mathfrak{F} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ then there exists a unique rule that associates to each smooth manifold M, each n-tuple of vector bundles $(E^1, \ldots, E^n)$ over M and each n-tuple of connections $(\nabla^1, \ldots, \nabla^n)$ on $(E^1, \ldots, E^n)$ respectively, a connection $\nabla = \mathfrak{F}(\nabla^1, \ldots, \nabla^n)$ on $\mathfrak{F}(E^1, \ldots, E^n)$ satisfying the following properties:*

(a) *(naturality with pull-backs) given smooth manifolds N, M and a smooth map $f : N \to M$ then*

$$f^*\big(\mathfrak{F}(\nabla^1, \ldots, \nabla^n)\big) = \mathfrak{F}(f^*\nabla^1, \ldots, f^*\nabla^n);$$

(b) *(naturality with morphisms) given vector bundle isomorphisms $L^i : E^i \to \widetilde{E}^i$, $i = 1, \ldots, n$, if ∇^i is a connection on E^i which is L^i-related with a connection $\widetilde{\nabla}^i$ on $\widetilde{E}^i$ then $\mathfrak{F}(\nabla^1, \ldots, \nabla^n)$ is $\mathfrak{F}(L^1, \ldots, L^n)$-related with $\mathfrak{F}(\widetilde{\nabla}^1, \ldots, \widetilde{\nabla}^n)$;*

(c) *given connections ∇^i and $\widetilde{\nabla}^i$ on E^i with $\nabla^i - \widetilde{\nabla}^i = \mathfrak{t}^i$, $i = 1, \ldots, n$, then:*

$$\mathfrak{F}(\nabla^1, \ldots, \nabla^n)_X s - \mathfrak{F}(\widetilde{\nabla}^1, \ldots, \widetilde{\nabla}^n)_X s = \overline{\mathfrak{F}}\big(\mathfrak{t}^1(X), \ldots, \mathfrak{t}^n(X)\big) s,$$

for all $s \in \Gamma\big(\mathfrak{F}(E^1, \ldots, E^n)\big)$;

(d) *(trivial bundle property) If* ∇^i *is the standard connection of the trivial bundle* $M \times E_0^i$ *then* $\mathfrak{F}(\nabla^1, \ldots, \nabla^n)$ *is the standard connection of the trivial bundle* $M \times \mathfrak{F}(E_0^1, \ldots, E_0^n)$.

Let $\mathfrak{F} = (\mathfrak{F}^1, \ldots, \mathfrak{F}^m)$ be an m-tuple of functors $\mathfrak{F}^i : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ and let $\mathfrak{G} : \underline{\mathfrak{Vec}}^m \to \underline{\mathfrak{Vec}}$ be a functor; we denote by $\mathfrak{G} \circ \mathfrak{F} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ the smooth functor defined[2] by:

$$(\mathfrak{G} \circ \mathfrak{F})(V_1, \ldots, V_n) = \mathfrak{G}\big(\mathfrak{F}^1(V_1, \ldots, V_n), \ldots \mathfrak{F}^m(V_1, \ldots, V_n)\big),$$

for all objects $V_1, \ldots, V_n$ of $\underline{\mathfrak{Vec}}$.

Proposition B.10. *Let* $\mathfrak{F} = (\mathfrak{F}^1, \ldots, \mathfrak{F}^m)$ *be an* m*-tuple of smooth functors* $\mathfrak{F}^i : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ *and let* $\mathfrak{G} : \underline{\mathfrak{Vec}}^m \to \underline{\mathfrak{Vec}}$ *be a smooth functor. Given vector bundles* $E^1, \ldots, E^n$ *over a smooth manifold* M *endowed respectively with connections* $\nabla^1, \ldots, \nabla^n$ *then:*

$$(\mathfrak{G} \circ \mathfrak{F})(\nabla^1, \ldots, \nabla^n) = \mathfrak{G}\big(\mathfrak{F}^1(\nabla^1, \ldots, \nabla^n), \ldots, \mathfrak{F}^m(\nabla^1, \ldots, \nabla^n)\big).$$

Moreover, if $\mathfrak{I} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ *denotes the identity functor of* $\underline{\mathfrak{Vec}}$ *then, given a connection* ∇ *on a vector bundle* E*, we have:*

$$\mathfrak{I}(\nabla) = \nabla.$$

Proposition B.11. *Given a smooth functor* $\mathfrak{F} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ *and smooth vector bundles* $\pi^i : E^i \to M$ *endowed with connections* ∇^i, $i = 1, \ldots, n$, *then the curvature tensor of the connection* $\mathfrak{F}(\nabla^1, \ldots, \nabla^n)$ *is given by:*

$$R_x(v, w) = \overline{\mathfrak{F}}\big(R_x^1(v, w), \ldots, R_x^n(v, w)\big),$$

for all $x \in M$, $v, w \in T_xM$, *where* R^i *denotes the curvature tensor of* ∇^i, $i = 1, \ldots, n$.

Definition B.12. Given a positive integer n and smooth functors $\mathfrak{F} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ and $\mathfrak{F}' : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ then a *smooth natural transformation* ρ from $\mathfrak{F}$ to $\mathfrak{F}'$ is a rule that associates to each object $(V_i)_{i=1}^n$ of $\underline{\mathfrak{Vec}}^n$ an open subset $A_{(V_1, \ldots, V_n)}$ of $\mathfrak{F}(V_1, \ldots, V_n)$ and a smooth map:

$$\rho_{V_1, \ldots, V_n} : A_{V_1, \ldots, V_n} \longrightarrow \mathfrak{F}'(V_1, \ldots, V_n)$$

such that, given objects $(V_i)_{i=1}^n$, $(W_i)_{i=1}^n$ of $\underline{\mathfrak{Vec}}^n$ and a morphism $(T_i)_{i=1}^n$ from $(V_i)_{i=1}^n$ to $(W_i)_{i=1}^n$ then $\mathfrak{F}(T_1, \ldots, T_n)$ carries $A_{V_1, \ldots, V_n}$ to $A_{W_1, \ldots, W_n}$

[2]We will usually only describe functors on objects; the action of the functor on morphisms should be clear.

and the diagram:

$$\begin{tikzcd}
A_{V_1,\ldots,V_n} \arrow[r, "\rho_{V_1,\ldots,V_n}"] \arrow[d, "\mathfrak{F}(T_1,\ldots,T_n)"'] & \mathfrak{F}'(V_1,\ldots,V_n) \arrow[d, "\mathfrak{F}'(T_1,\ldots,T_n)"] \\
A_{W_1,\ldots,W_n} \arrow[r, "\rho_{W_1,\ldots,W_n}"'] & \mathfrak{F}'(W_1,\ldots,W_n)
\end{tikzcd}$$

commutes.

Given a smooth natural transformation ρ from $\mathfrak{F}$ to $\mathfrak{F}'$ and given vector bundles $\pi^i : E^i \to M$, $i = 1,\ldots,n$, we obtain a fiber bundle morphism $\rho_{E^1,\ldots,E^n} : A \to \mathfrak{F}'(E^1,\ldots,E^n)$ defined on an open subset A of $\mathfrak{F}(E^1,\ldots,E^n)$ by setting:

$$A_x = A_{E^1_x,\ldots,E^n_x}, \quad (\rho_{E^1,\ldots,E^n})_x = \rho_{E^1_x,\ldots,E^n_x},$$

for all $x \in M$.

Proposition B.13. *Given a positive integer n, smooth functors $\mathfrak{F} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$, $\mathfrak{F}' : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$, a smooth natural transformation ρ from $\mathfrak{F}$ to $\mathfrak{F}'$, vector bundles $\pi^i : E^i \to M$ endowed with connections ∇^i, $i = 1,\ldots,n$, then:*

$$\mathfrak{F}'(\nabla^1,\ldots,\nabla^n)_v(\rho_{E^1,\ldots,E^n} \circ s) = \mathrm{d}\rho_{E^1_x,\ldots,E^n_x}\big(s(x)\big)\big(\mathfrak{F}(\nabla^1,\ldots,\nabla^n)_v s\big),$$

for all $x \in M$, $v \in T_xM$ and every smooth section s of $\mathfrak{F}(E^1,\ldots,E^n)$ with range contained in the domain of $\rho_{E^1,\ldots,E^n}$.

B.2. **Example.** Let n be a positive integer and consider the smooth functor $\mathfrak{S} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ defined by:

$$\mathfrak{S}(V_1,\ldots,V_n) = V_1 \oplus \cdots \oplus V_n.$$

Given vector bundles $E^1, \ldots, E^n$ over a smooth manifold M then $\mathfrak{S}(E^1,\ldots,E^n)$ is the Whitney sum of $E^1, \ldots, E^n$. Let ∇^i be a connection on E^i, $i = 1,\ldots,n$. For each $i = 1,\ldots,n$, consider the smooth functor $\mathfrak{P}^i : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ defined by:

$$\mathfrak{P}^i(V_1,\ldots,V_n) = V_i.$$

We have a smooth natural transformation ρ^i from $\mathfrak{S}$ to $\mathfrak{P}^i$ given by:

$$\rho^i : V_1 \oplus \cdots \oplus V_n \ni (v_1,\ldots,v_n) \longmapsto v_i \in V_i.$$

Set $\nabla = \mathfrak{S}(\nabla^1,\ldots,\nabla^n)$. Proposition B.13 implies that:

$$\nabla_v(s_1,\ldots,s_n) = (\nabla^1_v s_1,\ldots,\nabla^n_v s_n),$$

for all $s_1 \in \Gamma(E^1), \ldots, s_n \in \Gamma(E^n)$, $v \in TM$.

B.3. **Example.** Consider the smooth functors $\mathfrak{F} : \underline{\mathfrak{Vec}}^2 \to \underline{\mathfrak{Vec}}$ and $\mathfrak{G} : \underline{\mathfrak{Vec}}^2 \to \underline{\mathfrak{Vec}}$ defined as follows; let V_1, V_2, W_1, W_2 be objects of $\underline{\mathfrak{Vec}}$ and let $T_1 : V_1 \to W_1$, $T_2 : V_2 \to W_2$ be isomorphisms. We set:

$$\mathfrak{F}(V_1, V_2) = \mathrm{Lin}(V_1, V_2), \quad \mathfrak{F}(T_1, T_2)L = T_2 \circ L \circ T_1^{-1},$$
$$\mathfrak{G}(V_1, V_2) = \mathrm{Lin}(V_2^*, V_1^*) \quad \mathfrak{G}(T_1, T_2)R = (T_1^*)^{-1} \circ R^* \circ T_2^*,$$

for $L \in \mathrm{Lin}(V_1, V_2)$, $R \in \mathrm{Lin}(V_2^*, V_1^*)$. We have a natural transformation ρ from $\mathfrak{F}$ to $\mathfrak{G}$ defined by:

$$\rho : \mathrm{Lin}(V_1, V_2) \ni t \longmapsto t^* \in \mathrm{Lin}(V_2^*, V_1^*).$$

Let E^1, E^2 be vector bundles over a smooth manifold M endowed with connections ∇^1 and ∇^2 respectively. We denote by ∇ both the connections $\mathfrak{F}(\nabla^1, \nabla^2)$ and $\mathfrak{G}(\nabla^1, \nabla^2)$ on the bundles $\mathrm{Lin}(E^1, E^2)$ and $\mathrm{Lin}((E^2)^*, (E^1)^*)$ respectively. Proposition B.13 tells us that, given a smooth section L of $\mathrm{Lin}(E^1, E^2)$ then:

$$\nabla_v L^* = (\nabla_v L)^*,$$

for all $v \in TM$.

B.4. **Example.** Consider the smooth functor $\mathfrak{F} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ defined as follows; let V, W be objects of $\underline{\mathfrak{Vec}}$ and let $T : V \to W$ be an isomorphism. We set:

$$\mathfrak{F}(V) = \mathrm{Bilin}(V, V; V) \oplus V \oplus V,$$
$$\mathfrak{F}(T)(B, v_1, v_2) = \big(T \circ B(T^{-1}\cdot, T^{-1}\cdot), T(v_1), T(v_2)\big),$$

for every bilinear map $B : V \times V \to V$ and all $v_1, v_2 \in V$. We have a smooth natural transformation ρ from $\mathfrak{F}$ to the identity functor $\mathfrak{I}$ of $\underline{\mathfrak{Vec}}$ defined by:

$$\rho : \mathrm{Bilin}(V, V; V) \oplus V \oplus V \ni (B, v_1, v_2) \longmapsto B(v_1, v_2) \in V.$$

Let E be a vector bundle over a smooth manifold M endowed with a connection ∇. We will also denote by ∇ the connection $\mathfrak{F}(\nabla)$ on $\mathrm{Bilin}(E, E; E)$. Proposition B.13 tells us that, given a smooth section B of $\mathrm{Bilin}(E, E; E)$ and smooth sections s_1, s_2 of E then:

$$\nabla_v\big(B(s_1, s_2)\big) = (\nabla_v B)(s_1, s_2) + B(\nabla_v s_1, s_2) + B(s_1, \nabla_v s_2),$$

for all $v \in TM$.

B.5. **Example.** Consider the smooth functor $\mathfrak{F} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ defined as follows; let V, W be objects of $\underline{\mathfrak{Vec}}$ and let $T : V \to W$ be an isomorphism. We set:

$$\mathfrak{F}(V) = \mathrm{Lin}(V),$$
$$\mathfrak{F}(T)L = T \circ L \circ T^{-1},$$

for all $L \in \mathrm{Lin}(V)$. We have a smooth natural transformation ρ from $\mathfrak{F}$ to itself defined by:

$$\rho : \mathrm{Lin}(V) \supset \mathrm{GL}(V) \ni L \longmapsto L^{-1} \in \mathrm{Lin}(V).$$

Let E be a vector bundle over a smooth manifold M endowed with a connection ∇. We will also denote by ∇ the connection $\mathfrak{F}(\nabla)$ on $\mathrm{Lin}(E)$. Let L be a smooth section of $\mathrm{Lin}(E)$ such that L_x is an isomorphism of E_x, for all $x \in M$. Proposition B.13 tells us that:

$$\nabla_v(L^{-1}) = -L^{-1}(\nabla_v L)L^{-1},$$

for all $v \in TM$.

References

[1] R. A. Blumenthal and J. J. Hebda, The generalized Cartan–Ambrose–Hicks theorem, *Geom. Dedicata* **29** (1989), 163–175.

[2] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol I*, Interscience Publishers, New York-London, 1963.

[3] S. Lang, *Fundamentals of differential geometry*, Graduate Texts in Mathematics. 191. New York, NY: Springer, 2001.

[4] F. Nübel, On integral manifolds for vector space distributions, *Math. Ann.* **294** (1992), 1–17.

[5] K. Pawel and H. Reckziegel, Affine submanifolds and the theorem of Cartan–Ambrose–Hicks, *Kodai Math. J.* **25** (2002), 341–356.

[6] P. Piccione and D. V. Tausk, *Connections compatible with tensors. A characterization of left-invariant Levi–Civita connections in Lie groups*, arXiv `math.DG/0509656`.

[7] H. J. Sussmann, Orbits of families of vector fields and integrability of distributions, *Trans. Am. Math. Soc.* **180** (1973), 171–188.

[8] J. A. Wolf, *Spaces of constant curvature*. New York, Mc-Graw-Hill, 1967.

Departamento de Matemática, Universidade de São Paulo, Brazil

E-mail address: `piccione@ime.usp.br, tausk@ime.usp.br`

URL: `http://www.ime.usp.br/~piccione,`
`http://www.ime.usp.br/~tausk`