Resenhas IME-USP 2001, Vol. 5, No. 2, 149 - 158.

Free Akivis algebras, primitive elements, and hyperalgebras [1]

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Abstract: Free Akivis algebras and primitive elements in their universal enveloping algebras are investigated. The conjecture of K .H . Hofmann and K. Strambach on the structure of primitive elements is proved to be not valid, and a full system of primitive elements in free nonassociative algebra is contructed. It is proved that every algebra can be considered as a hyperalgebra, that is, a system with a series of multilinear operations that plays a role of a tangent algebra for a local analytic loop, where the hyperalgebra operations on B are interpreted by certain primitive elements.

Key words: Akivis algebra, universal enveloping algebra, primitive element, local analytic loop, hyperalgebra.

1 Introduction

A vector space A is called an *Akivis algebra* if it is endowed with two operations: an anticommutative bilinear operation $[x, y]$, (*a commutator*), and a trilinear operation $\mathcal{A}(x, y, z)$ (*an associator*), that are related by means of the identity

$$\begin{aligned}[[x,y],z]+[[y,z],x]+[[z,x],y] = {} & \mathcal{A}(x,y,z)+\mathcal{A}(y,z,x)+\mathcal{A}(z,x,y) \\ & -\mathcal{A}(y,x,z)-\mathcal{A}(x,z,y)-\mathcal{A}(z,y,x). \qquad (1)\end{aligned}$$

These algebras were introduced in 1976 by M. A. Akivis [1], under the name *W-algebras*, as local algebras of three-webs (or of local analytic loops).

Let L be a local analytic loop with the multiplication $x \cdot y$, left division $y \backslash x$, and right division x/y (see, for example, [4]). The tangent space of L at the unit 0 may be identified with L itself; and one may endow this space with the following two operations that represent the deviation from commutativity and from associativity of the multiplication $x \cdot y$ in the loop L:

$$\begin{aligned}[x,y] &= \lim_{t\mapsto 0} t^{-2}\left((tx\cdot ty)\,/\,(ty\cdot tx)\right) = \\ (&= \lim_{t\mapsto 0} t^{-2}\left((ty\cdot tx)\setminus(tx\cdot ty)\right) \\ &= \lim_{t\mapsto 0} t^{-2}\left(tx\cdot ty - ty\cdot tx\right)\ ), \\ \mathcal{A}(x,y,z) &= \lim_{t\mapsto 0} t^{-3}\left(((tx\cdot ty)\cdot tz)\,/\,(tx\cdot(ty\cdot tz))\right)\end{aligned}$$

[1]The extended version of the lecture presented at the IX Encontro em Álgebra USP/UNICAMP, September 19–22, 2001, São Pedro, SP, Brazil. Supported by the CNPq grant 300528/99-0, FAPESP Proc. 2001/06608-3 and FAPESP Proc.00/06832-8

$$\begin{aligned} (&= \lim_{t\mapsto 0} t^{-3}\left((tx\cdot(ty\cdot tz))\setminus((tx\cdot ty)\cdot tz)\right) \\ &= \lim_{t\mapsto 0} t^{-3}\left((tx\cdot ty)\cdot tz - tx\cdot(ty\cdot tz)\right)\ ), \end{aligned}$$

where x, y, z are vectors from the tangent space, $t \in \mathbf{R}$. It was proved in [1] that with respect to these operations the tangent space of the loop L forms an Akivis algebra. We will denote this Akivis algebra as $\mathcal{A}(L)$.

If a loop L is associative, i.e., L is a Lie group, then the operation $\mathcal{A}(x, y, z)$ is trivial, and so the Akivis identity (1) converts to the well known Jacobi identity. Hence, in this case the algebra $\mathcal{A}(L)$ is a Lie algebra. If L satisfies the Moufang identity

$$(xy)(zx) = x(yz\cdot x),$$

then the function $\mathcal{A}(x, y, z)$ becomes skewsymmetric [1] and so by (1) this function can be represented in terms of the bilinear operation $[x, y]$:

$$\mathcal{A}(x, y, z) = 1/6J(x, y, z),$$

where $J(x, y, z) = [[x, y], z] + [[z, x], y] + [[y, z], x]$ is the *Jacobian* of the elements x, y, z. Moreover, in this case $\mathcal{A}(L)$ satisfies the following *Malcev identity*

$$[J(x, y, z), x] = J(x, y, [x, z]),$$

hence $\mathcal{A}(L)$ is a *Malcev algebra* (see [8]). In general case the operation $\mathcal{A}(x, y, z)$ is not expressed in terms of the commutator $[x, y]$.

Now, let B be a (not necessary associative) algebra with a bilinear multiplication $(x, y) \mapsto xy$. Consider in B the usual *commutator* $[x, y] = xy - yx$ and *associator* $\mathcal{A}(x, y, z) = (xy)z - x(yz)$ functions; then it is easily checked that these functions satisfy identity (1). Hence B is an Akivis algebra with respect to these operations. We will denote this algebra by $Ak\,(B)$.

It was conjectured by M. A. Akivis [1] (see also [2, Problem X.3.8], [4, Problem IX.6.12]) and proved by I. P. Shestakov [11, 12] that every Akivis algebra A can be isomorphically embedded into an Akivis algebra $Ak\,(B)$ for a suitable algebra B. Moreover, a basis of the universal enveloping algebra $U(A)$ was constructed in [11, 12].

In the present paper, we give some new results on the structure of free Akivis algebras and primitive elements in their universal enveloping algebras. We claim that subalgebras of free Akivis algebras are free and that finitely generated subalgebras are finitely residual. Decidability of the word problem for the variety of Akivis algebras is also proclaimed.

The conjecture of K.H.Hofmann and K.Strambach [4, Problem 6.15] on the structure of primitive elements is proved to be not valid, and a full system of primitive elements in free nonassociative algebra is constructed.

Finally, we show that every algebra B can be considered as a hyperalgebra, that is, a system with a series of multilinear operations that plays a role of a tangent algebra for a local analytic loop, where the hyperalgebra operations on B are interpreted by certain primitive elements.

The full proofs of the presented results will appear in [13].

2 Free Akivis algebras

Let A be an Akivis algebra over a field F with a linear basis

$$e_1, e_2, \dots e_\alpha, \dots.$$

Consider the set of words

$$V = \{e_i, e_i e_j, (e_i e_j)e_k | i \le j \le k\}$$

in the universal enveloping algebra $U(A)$ (see [11, 12]). Set also $|e_1| = 1$, $|e_i e_j| = 2$, $|(e_i e_j)e_k| = 3$. Denote by V^* the set of all nonassociative words in the alphabet V, including the unit 1 considered as the empty word, and by V^0 the set of all words from V^* that do not contain the subwords of type $v_1 v_2$, where $v_1, v_2 \in V$, $|v_1| + |v_2| \le 3$. The elements of V^0 are called v^0-words in the alphabet $e_1, e_2, \dots e_\alpha, \dots$. The first author in [11, 12] proved that the v^0-words form a basis of the algebra $U(A)$.

For an algebra C and a subset $M \subseteq C$, we denote by $alg_C\langle M\rangle$ and $idl_C\langle M\rangle$ the subalgebra and the ideal of C generated by M.

Lemma 1 *Let A be an Akivis algebra and M be a subset of A. Then the following statements are true:*
1) $alg_A\langle M\rangle = alg_{U(A)}\langle M\rangle \cap A$;
2) $idl_A\langle M\rangle = idl_{U(A)}\langle M\rangle \cap A$.

Remind that a variety of algebras is called *Schreier* if every subalgebra of a free algebra in this variety is also free. The general properties of Schreier varieties were investigated in [7, 17]. The well-known examples of Schreier varieties are the varieties of all nonassociative algebras [6], commutative and anticommutative algebras [15], Lie algebras [14, 18] and Lie superalgebras [10, 16].

Using lemma 1 and the results of [17], we prove

Theorem 1 *The variety of Akivis algebras is Schreier.*

Theorem 1 has two standard corollaries [7].

Corollary 1 *Automorphisms of finitely generated free Akivis algebras are tame.*

Corollary 2 *The occurrence problem for free Akivis algebras is decidable.*

Furthermore, applying the corresponding results for the variety of all nonassociative algebras (see [5, 20]), one can prove the following theorems.

Theorem 2 *Finitely generated subalgebras of free Akivis algebras are residually finite.*

Theorem 3 *Word problem is decidable for variety of Akivis algebras.*

3 Primitive elements of nonassociative algebras

Let A be an Akivis algebra over a field F and $U(A)$ its universal enveloping algebra. It is easily checked that the linear mapping

$$\Delta : A \longrightarrow Ak(U(A) \otimes_F U(A))$$

given by the rule

$$\Delta(u) = u \otimes 1 + 1 \otimes u, \ u \in A, \tag{2}$$

is an embedding of Akivis algebras.

By definition of the universal enveloping algebra $U(A)$, (see [12]), the homomorphism Δ can be uniquely extended to the homomorphism

$$\Delta : U(A) \longrightarrow U(A) \otimes_F U(A).$$

An element $u \in U(A)$ is called *primitive* if it satisfies equality (2).

One can easily check that the set $P(A)$ of all primitive elements of the algebra $U(A)$ is closed under the operations $[x, y], (x, y, z)$; that is, $P(A)$ is an Akivis subalgebra of $Ak(U(A))$. By definition of Δ, every element from A is primitive. K. H. Hofmann and K. Strambach formulated in [4, Problem 6.15] the question about validity of the equality $A = P(A)$, for algebras over fields of characteristic 0. Note that in the case of Lie algebras this equality turns to the well-known Friedrichs criterion (see [3]) for Lie elements in universal enveloping algebras.

We first show that the question of K. H. Hofmann and K. Strambach is answered negatively. Let A be a free Akivis algebra on a single free generator x. Then $U(A) = F\{x\}$ is a free nonassociative algebra generated by x. Consider the element

$$f = (x^2, x, x) - 2x(x, x, x) = (x^2x)x - x^2x^2 - 2x(x, x, x).$$

The element f does not belong to A since it is a linear combination of v^0-words in the alphabet $x, (x, x, x)$, with length ≥ 2. On the other hand, one can straightforwardly check that equation (2) is true for f, that is, $f \in P(A)$.

This example shows that the operations $[x, y], (x, y, z)$ do not produce all the primitive elements in $U(A)$. The remainder of this section will be devoted to construction of a full system of primitive operations on nonassociative algebras.

Let B be the free nonassociative algebra on the set of free generators $X \cup Y \cup \{z\}$, where $X = \{x_1, \ldots, x_n, \ldots\}$, $Y = \{y_1, \ldots, y_n, \ldots\}$. Then $B = U(A)$, where A is the free Akivis algebra on the same set of generators, and so we may consider primitive elements in B.

Denote by X^1 the set of all right normed words of the type

$$u = (\ldots(x_{i_1}x_{i_2})\ldots)x_{i_m} = x_{i_1}R_{x_{i_2}}\ldots R_{x_{i_m}}, \tag{3}$$

where $i_1 < i_2 < \ldots < i_m$, $m \geq 0$. If u is a word of type (3), then we put $|u| = m$, $supp\,(u) = \{i_1, i_2, \ldots, i_m\}$. Besides, we will use the following simplified notation for this word:

$$u = x_{i_1} x_{i_2} \ldots x_{i_m},$$

omitting the parenthesis. For any words $u, v \in X^1$, we set $u \leq v$ if $supp\,(u) \subseteq supp\,(v)$. If $u \leq v$ and $u \neq v$, we write $u < v$. In particular, $u > 1$ for any u of type (3) with $|u| \geq 1$.

An ordered sequence $(u_1, u_2, \ldots, u_k)$ of elements $u_i \in X^1$, $1 \leq i \leq k$, is called a *k-decomposition* of the word $u \in X^1$ if $supp\,(u_i) \cap supp\,(u_j) = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{n} supp\,(u_i) = supp\,(u)$. For a given word $u \in X^1$, we will denote by $\sum_{u_1,u_2}$ (or $\sum_{u_1,u_2,u_3}$) the sum over all the 2-decompositions (or correspondently, 3-decompositions) of the word u. Moreover, we will omit indexes u_1, u_2 in this notation if it is clear from the context what kind of decomposition is under consideration.

Similarly, we define the set Y^1 and extend to it all the previous notations. In the sequel, if the parenthesis are not arranged in a product then the product is considered as a right normed one.

Let $u = x_1 x_2 \ldots x_m \in X^1$, $v = y_1 y_2 \ldots y_n \in Y^1$. By induction on $m + n = k$, where $m, n \geq 1$, we define polynomials

$$p_{m,n}(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n; z) = p_{m,n}(u, v, z) \in B$$

in the following way.

1. $p_{1,1}(x_1; y_1; z) = (x_1, y_1, z)$;

2. Suppose that for all $m, n \geq 1$ and $m + n < k$, $k \geq 3$, the polynomials $p_{m,n}(u, v, z)$ are already defined.

3. Let $m + n = k \geq 3$. Put

$$p_{m,n}(u, v, z) = (u, v, z) - \sum_{|u_1|+|v_1|\geq 1,\, u_2, v_2 > 1} u_1 v_1 p_{m-|u_1|, n-|v_1|}(u_2, v_2, z).$$

Since the numbers $m = |u|, n = |v|$ are uniquely defined by u, v, we reduce the notation $p_{m,n}(u, v, z)$ to $p(u, v, z)$. To simplify the formulas below, we extend the definition of polynomials $p_{m,n}(u, v, z)$ also for the cases $m = 0$ or $n = 0$, by setting $p_{0,n}(1, v, z) = p_{m,0}(u, 1, z) = 0$. With this notations, we have the equality

$$(u, v, z) = \sum u_1 v_1 p(u_2, v_2, z). \tag{4}$$

For $f, g \in B$ we denote $f \circ g = f \otimes g + g \otimes f \in B \otimes_F B$. We have the homomorphism

$$\Delta : B \longrightarrow B \otimes_F B,$$

defined by the rules

$$\Delta(x_i) = 1 \circ x_i,\ \Delta(y_i) = 1 \circ y_i,\ \Delta(z) = 1 \circ z,\ i \geq 1.$$

By definition, an element $f \in B$ is primitive if and only if $\Delta(f) = 1 \circ f$.

Lemma 2 *Let $u = x_1x_2\dots x_m \in X^1,\ v = y_1y_2\dots y_n \in Y^1$. Then the following equalities hold:*

1) $\Delta(u) = \sum u_1 \otimes u_2$;
2) $\Delta(uv) = \sum u_1v_1 \otimes u_2v_2$;
3) $\Delta(uvz) = \sum u_1v_1 \circ (u_2v_2z)$;
4) $\Delta(u(vz)) = \sum u_1v_1 \circ (u_2(v_2z))$;
5) $\Delta(u, v, z) = \sum u_1v_1 \circ (u_2, v_2, z)$.

Using the lemma, we prove that all the elements of type $p(u, v, z)$ are primitive.

Theorem 4 *Let $u = x_1x_2\dots x_m \in X^1,\ v = y_1y_2\dots y_n \in Y^1,\ m, n \geq 1$. Then $p_{m,n}(u, v, z)$ is a primitive element of B.*

Corollary 3 *Let C be an algebra with the unit 1 and assume that there exists an algebra homomorphism $\delta : C \to C \otimes_F C$. Then the space $Prim\,(C, \delta) = \{p \in C | \delta(p) = 1 \circ p\}$ of δ-primitive elements of the algebra C is closed under the operations*

$$[x, y],\ p_{m,n}(x_1, x_2, \dots, x_m; y_1, y_2, \dots, y_n; z),\ m, n \geq 1. \tag{5}$$

Notice that if F is a field of characteristic $p > 0$, then the operation x^p can be added to the set of operations (5).

The following theorem shows completeness of the set of primitive operations (5).

Theorem 5 *Let C be an algebra with the unit 1 over a field of characteristic 0, and $\delta : C \to C \otimes_F C$ be a nontrivial homomorphism of algebras. Suppose that the algebra C is generated by a set M of δ-primitive elements, and let $P(M)$ be the minimal subspace of C that contains M and is closed with respect to primitive operations (5). Let $e_1, e_2, \dots, e_\alpha, \dots$ be a basis of $P(M)$. Then the set of right normed words of the type*

$$e_{i_1}e_{i_2}\dots e_{i_k}, \tag{6}$$

where $i_1 \leq i_2 \leq \dots \leq i_k,\ k \geq 0$, forms a basis of the algebra C.

Corollary 4 *Under the assumptions of the theorem, the set $P(M)$ coincides with the set $Prim\,(C, \delta)$ of all δ-primitive elements of C. In other words, any set of δ-primitive elements which generates C generates also the set $Prim\,(C, \delta)$ by operations (5).*

4 Hyperalgebras and primitive elements

Let F be a field of characteristic 0. If L is a Lie algebra, then by the Friedrichs criterion [3], the set of primitive elements of the algebra $U(L)$ coincides with L. Note that L is closed under the primitive operation $[x, y]$, and all the other primitive operations $p_{m,n}$ are identically zero on L.

Lie algebras first appeared as tangent algebras of Lie groups and, in case of simply connected Lie groups, they determine the corresponding groups up to isomorphism. It is known [4] that Akivis algebras do not determine, in general, the corresponding local analytic loops. It was shown by P.O.Miheev and L.V.Sabinin [9] that a simply connected local analytic loop is determined up to isomorphism by a more sophisticated analogue of tangent algebra, the so called hyperalgebra, an algebraic system with a series of multilinear operations. It is natural to ask whether the operations in hyperalgebras can be interpreted by primitive elements in a nonassociative algebra. Below we give an interpretation of hyperalgebra operations in nonassociative algebras.

First, let us remind the definition of a hyperalgebra.

A vector space A over a field F is called a *hyperalgebra* if it is endowed with the multilinear operations

$$< x_1, x_2, \ldots, x_m, y, z >, \ m \geq 0,$$
$$\Phi(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n), \ m \geq 1, n \geq 2,$$

which satisfy the identities:

$$< x_1, x_2, \ldots, x_m, y, z >= - < x_1, x_2, \ldots, x_m, z, y >, \tag{7}$$

$$< x_1, x_2, \ldots, x_r, a, b, x_{r+1}, \ldots, x_m, y, z > - < x_1, x_2, \ldots, x_r, b, a, x_{r+1}, \ldots, x_m, y, z >$$
$$+ \sum_{k=0} \sum_{\alpha} < x_{\alpha_1}, \ldots, x_{\alpha_k}, < x_{\alpha_{k+1}}, \ldots, x_{\alpha_r}, a, b >, x_{r+1}, \ldots, x_m, y, z >= 0, \tag{8}$$

$$\sigma_{x,y,z}(< x_1, \ldots, x_r, x, y, z > +$$
$$\sum_{k=0}^{r} \sum_{\alpha} < x_{\alpha_1}, \ldots, x_{\alpha_k}, < x_{\alpha_{k+1}}, \ldots, x_{\alpha_r}, y, z >, x >) = 0, \tag{9}$$

$$\Phi(x_1, \ldots, x_m, y_1, \ldots, y_n) = \Phi(x_{\tau(1)}, \ldots, x_{\tau(m)}, y_{\delta(1)}, \ldots, y_{\delta(n)}), \tag{10}$$

where α runs the set of all bijections of the type $\alpha : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, r\}$, $i \mapsto \alpha_i$, $\alpha_1 < \alpha_2 < \ldots < \alpha_k$, $\alpha_{k+1} < \ldots < \alpha_r$, $k = 0, 1, \ldots, r$, $r \geq 0$, $\sigma_{x,y,z}$ denotes the cyclic sum by x, y, z; $\tau \in S_m$, $\delta \in S_n$, S_l is the symmetric group.

Let B be a (nonassociative) algebra over a field F of characteristic 0. For any $x_1, x_2, \ldots, x_m, y, z \in B$ we put

$$< y, z > \ = \ -[y, z] = < 1, y, z >, \tag{11}$$

$$< x_1, x_2, \ldots, x_m, y, z > \ = \ -p_{m,1}(u, y, z) + p_{m,1}(u, z, y) = < u, y, z >, \tag{12}$$

where $u = x_1 x_2 \ldots x_m,\ m \geq 1$.

If $m \geq 1, n \geq 2$, then for any $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ we put

$$\Phi(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = \\ = \frac{1}{m!}\frac{1}{n!} \sum_{\tau \in S_m, \delta \in S_n} p_{m,n-1}(x_{\tau(1)}, \ldots, x_{\tau(m)}; y_{\delta(1)}, \ldots; y_{\delta(n)}). \tag{13}$$

By (4), we have

$$(u, z, y) - (u, y, z) = \sum_{u_2 > 1} u_1 < u_2, y, z > . \tag{14}$$

Note that definition (12) does not give (11) when $m = 0$, because $p_{0,1}(1, y, z) = 0$. The restriction $u_2 > 1$ in (14) is caused by this fact and means that the summand with $< 1, y, z >$ does not appear there.

Observe also that if $x_1 = x_2 = \ldots = x_m = x,\ y_1 = y_2 = \ldots = y_n = y$, then

$$\Phi(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = p_{m,n-1}(x^m, y^{n-1}, y).$$

Denote by $G(B)$ the space B considered as an algebra under operations (11), (12), (13).

Theorem 6 *$G(B)$ is a hyperalgebra.*

It is natural to ask the following question.

Problem 1 *Is it true that any hyperalgebra can be isomorphically embedded into a hyperalgebra $G(B)$ for a suitable algebra B?*

In case of affirmative solution of this problem every hyperalgebra would have a universal enveloping algebra with the Poincare-Birkhoff-Witt basis as in theorem 5. Remind that, in particular, Lie algebras, Malcev algebras, Bol algebras are hyperalgebras (see [9]). All the hyperalgebra operations except $[x, y]$ are trivial in Lie algebras. Operations $\Phi(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$ are trivial in Malcev and Bol algebras.

Note that the definition of hyperalgebras given above was motivated by and oriented to a description of identities of tangent algebras of right-monoalternative local analytic loops (see [9]). It would be interesting to find an alternative definition of hyperalgebras, probably, with another operations, that is more closely related to primitive operations (5). Maybe, this could help to solve problem 1.

Another natural question that related closely to problem 1 is the following.

Problem 2 *To give an intrinsic characterization of the set Prim (B) of primitive elements in a free nonassociative algebra B, in terms of primitive operations (5).*

In particular, is it possible to choose a definition of hyperalgebra in such a way that *Prim* (B) would be a hyperalgebra with respect to operations (5) or, probably, relatively to another set of primitive operations?

References

[1] M. A. Akivis, The local algebras of a multidimensional three-web (Russian), Sibirsk. Mat. Z. 17, No 1 (1976), 5–11. English translation: Siberian Math. J. 17 (1976), No 1, 3–8.

[2] V. V. Goldberg, Local differentiable quasigroups ans webs, in "Quasigroups and Loops Theory and Applications", ed. O.Chein, H.O.Pflugfelder, J.D.H.Smith, Sigma Series in Pure Mathematics, v.8, Heldermann Verlag, Berlin, 1990, 263–311.

[3] N. Jacobson, Lie algebras, Dover Publications, Inc, N.Y., 1979.

[4] K. H. Hofmann, K. Strambach, Topological and analytic loops, in "Quasigroups and Loops Theory and Applications", ed. O.Chein, H.O.Pflugfelder, J.D.H.Smith, Sigma Series in Pure Mathematics, v.8, Heldermann Verlag, Berlin, 1990, 205–262.

[5] G. V. Kryazhovskikh, Approksimability of finitely presented algebras (Russian), Sibirsk. Mat. Z. 21, No 5 (1980), 58–62. English translation: Siberian Math. J. 21 (1980), No 5, 688–691.

[6] A. G. Kurosh, Nonassociative free algebras and free products of algebras (Russian), Mat. Sb. 20, (1947), 119–126.

[7] J. Lewin, On Schreier varieties of linear algebras, Trans. Amer. Math. Soc. 132(1968), 553–582.

[8] A. I. Malcev, Analytic loops (Russian), Mat. sb. (N.S.) 36(78), no.3 (1955), 569–573.

[9] P. O. Miheev, L. V. Sabinin, On the infinitesimal theory of local analytic loops (Russian), Dokl. Akad. Nauk SSSR 297, No 4 (1987), 801–804. English translation: Soviet Math. Dokl. 36, No 3 (1988), 545–548.

[10] A. A. Mikhalev, Subalgebras of free colour Lie superalgebras (Russian), Mat. Zametki 37, No 5 (1985), 653–661.

[11] I. P. Shestakov, Linear representability of Akivis algebras (Russian), Dokl. Akad. Nauk 368, No 1 (1999), 21–23.

[12] I. P. Shestakov, Every Akivis algebra is linear, Geometriae Dedicata 77(1999), 215–223.

[13] I. P. Shestakov, U. U. Umirbaev, Free Akivis algebras, primitive elements, and hyperalgebras, J.of Algebra, to appear.

[14] A. I. Shirshov, Subalgebras of free Lie algebras (Russian), Mat. Sb. 33, No 2 (1953), 441–452.

[15] A. I. Shirshov, Subalgebras of free commutative and free anticommutative algebras (Russian), Mat. Sb. 34, No 1 (1954), 81–88.

[16] A. S. Shtern, Free Lie superalgebras (Russian), Sibirsk. Mat. Z. 27, No 1 (1986), 170–174.

[17] U. U. Umirbaev, Schreier varieties of algebras (Russian), Algebra i logika 33, No 3 (1994), 317–340. English translation: Algebra and Logic 33, No 3 (1994), 180–193.

[18] E. Witt, Die Unterringe der freien Lieschen ring, Mat. Z. 64 (1956), 195–216.

[19] K. A. Zhevlakov, A. M. Slinko, I. P. Shestakov, A. I. Shirshov, Rings that are nearly associative, Academic Press, N.Y., 1982.

[20] A. I. Zhukov, Reduced systems of defining relations in nonassociative algebras (Russian), Mat. Sb. 27 (1950), 267–280.

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