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An Introduction to Köthe's Conjecture and Polynomial Rings [1]

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Abstract: In this paper we give an introduction to Köthe's conjecture. We summarize results on the conjecture emphasizing relations with polynomial rings. We also give some results on polynomial rings in several indeterminates and connected questions.

Key words: Köthe's conjecture, nil ideals, nil radical, polynomial rings.

Introduction

The most famous open problem in the theory of associative nil rings is Köthe's problem, known as Köthe's conjecture. It asks whether the two-sided ideal generated by left nil ideals must be nil [12]. This question and many other related questions have extensively been studied. Recently several new interesting results were obtained and very good surveys have been written [18, 25].

The purpose of this paper is the following. First we give an introduction to the problem and present several equivalences. One of this equivalence concerns with polynomial rings. Next we present some new results and questions on polynomial rings in several indeterminates obtained in a paper by Wisbauer and the author [6] which are connected to the conjecture. Finally, a list of references on the subject and related problems are included. The reader will find this list very useful and should also take a look to the references in [18, 25].

All rings considered here are associative but do not necessarily have an identity element. The Jacobson radical of R will be denoted by $J(R)$. For background on Ring Theory the reader may see [15].

Let R be a ring. For an (right, left, two-sided) ideal I of R and integer $n \geq 1$, I^n denotes the (right, left, two-sided) ideal of R generated by all the products $x_1 \dots x_n$, where $x_i \in I$ for any i. Recall that an element $a \in R$ is said to be nilpotent if there exists an integer $n \geq 1$ such that $a^n = 0$. An (right, left, two-sided) ideal I of R is said to be nil if any element of I is nilpotent and is said to be nilpotent if $I^n = 0$, for some $n \geq 1$.

It is clear that if R is a commutative ring, then any nilpotent element of R is contained in a nilpotent ideal. Thus the sum of all nil ideals coincides with the

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sum of all nilpotent ideals and is equal to the set of all nilpotent elements of R. This ideal is called the nil radical of R and will be denoted here by $Nil(R)$. It is well-known that this is no more true when R is non-commutative. In general, the (upper) nil radical $Nil(R)$ of R is defined as the sum of all the nil ideals of R.

If I and J are (two-sided) nil (nilpotent) ideals of R, then $I + J$ is also nil (nilpotent). Furthermore, if I and J are right (left) nilpotent ideals of R, then $I + J$ is a nilpotent right (left) ideal too. However, it is not known whether the sum of two nil right (left) ideals of R is also nil. This is the content of the celebrated Köthe's conjecture.

1 Köthe's Problem

The so called Köthe's conjecture was stated for the first time by Köthe in 1930 [12]. The problem can be presented in several elementary equivalent ways and there are many other equivalent formulations. This problem is still open and seems to be unsolvable. We begin with the following formulation.

Lemma 1.1 *The following conditions are equivalent:*

(i) *If R is any ring and I is a nil right (left) ideal of R, there exists a two-sided nil ideal H such that $I \subseteq H$.*

(ii) *If R is any ring and I, J are nil right (left) ideals of R, then $I + J$ is also nil.*

(iii) *If R is a ring with $Nil(R) = 0$, then R do not have nil one-sided ideals.*

(iv) *If R is any ring and I is a nil right (left) ideal of R, then $I \subseteq Nil(R)$.*

Köthe's Problem *We say that Köthe's problem (briefly (KP)) has a positive solution if the equivalent conditions of Lemma 1.1 are satisfied.*

There are many assertions equivalent to Köthe's conjecture. Now we present some of them. First, Krempa in 1972 [13] and independently Sands in 1973 [23] proved the next result.

Theorem 1.2 *The following statements are equivalent:*

(i) *(KP) has a positive solution.*

(ii) *For every nil ring R the ring of 2×2 matrices over R is nil.*

(iii) *For every nil ring R the ring of $n \times n$ matrices over R is nil.*

Recall that the nil radical of a matrix ring $M_n(R)$ is equal to $M_n(I)$, for an ideal I of R. Hence it is easy to show that another equivalent formulation of condition (iii) (resp. (ii)) in Theorem 1.2 is the following: for any ring R and any $n \geq 2$ (resp. $n = 2$), the nil radical of $M_n(R)$ is equal to $M_n(Nil(R))$.

There is another result by Fisher and Krempa [7] appeared in 1983 which has similar mood as the above results. In fact, assume that R is a ring and G is a finite group of automorphisms of R. Then R is said to have no additive $\mid G \mid$-torsion if $\mid G \mid x = 0$ implies $x = 0$, for any $x \in R$, where $\mid G \mid$ is the order of G. Let R^G denote the set of all the elements $x \in R$ which are fixed under G, i.e., such that $x^g = x$, for all $g \in G$. The trace of $x \in R$ is defined by $tr(x) = \sum_{g \in G} x^g$. Clearly $tr(x) \in R^G$. We have

Theorem 1.3 ([7]) *The following conditions are equivalent:*

(i) *(KP) has a positive solution.*

(ii) *For any finite group G of automorphisms of a ring R such that R has no additive $\mid G \mid$-torsion, if R^G is nil, then R is nil.*

(iii) *For any finite group G of automorphisms of a ring R such that R has no additive $\mid G \mid$-torsion, $Nil(R^G) = Nil(R) \cap R^G$.*

(iv) *For any finite group G of automorphisms of a ring R such that R has no additive $\mid G \mid$-torsion, if $tr(R)$ is nil, then R is nil.*

Another equivalent formulation was obtained by Puczyłowski and the author. Assume that R is a ring which is a sum $R = R_1 + R_2$ of two subrings R_1 and R_2. In 1962 Kegel [11] proved that if both subrings $R_i, i = 1, 2$, are nilpotent, then R is nilpotent. The similar question for other radical properties was the motivation of [5]. Among other results we proved

Theorem 1.4 *(KP) has a positive solution if and only if when R_1 is nilpotent and R_2 is nil, then R is nil, where $R = R_1 + R_2$ as above.*

A ring R is said to be right T-nilpotent if for every sequence $x_1, x_2, \ldots$ of elements of R there exists an n such that $x_n \ldots x_1 = 0$. In [5] we also proved that (KP) has a positive solution if and only if when R_1 is right (left) T-nilpotent and R_2 is nil, then R is nil.

We can mention some results giving a positive answer to Köthe's conjecture in some special classes of rings. For example, if R is a right (left) artinian ring every nil right (left) ideal of R is nilpotent [15]. Thus (KP) has a positive solution in the class of right (left) artinian rings.

A ring R is said to be locally nilpotent if every finitely generated subring of R is nilpotent. If I and J are right (left) ideals of R which are locally nilpotent, then $I + J$ is locally nilpotent. Consequently, since for rings which are either right (left) noetherian or satisfy a polynomial identity the nil radical coincides with the locally nilpotent radical ([15], [21]), then (KP) has a positive solution in these classes of rings.

Another result which may help to find a positive solution to (KP) was obtained in [13]. From this result we see that to obtain a solution to (KP) it would be enough to find a solution to the problem in the class of algebras over fields.

In fact, he proved

Theorem 1.5 *(KP) has a positive solution if and only if it has a positive solution for algebras over any field.*

On the other hand, in [2] Amitsur proved that the Jacobson radical of infinitely generated algebras over uncontable fields is nil. As a consequence he got a positive answer to (KP) for algebras over uncontable fields. He asked whether the Jacobson radical of all finitely generated algebras is nil. However a negative solution of this problem was obtained in [3].

2 Köthe's Problem and Polynomial Rings

In 1956 Amitsur [1] proved that the Jacobson radical $J = J(R[x])$ of a polynomial ring $R[x]$ in one indeterminate x is equal to $N[x]$, where $N = J \cap R$ is a nil ideal of R. The question on whether N is equal to the nil radical of R remained an open problem. In a paper quoted in Section 1 Krempa proved also that another formulation for Köthe's conjecture is an affirmative answer for this question.

Recall that a ring R is said to be Jacobson radical if it cannot be homomorphically mapped onto a (right, left) primitive ring. We have the following

Theorem 2.1 *[13] (KP) has a positive solution if and only if for every nil ring R the polynomial ring $R[x]$ in one indeterminate x is Jacobson radical.*

It is clear that this is equivalent to the open problem on whether $N = Nil(R)$ mentioned above.

Consequently, to solve (KP) it would be enough to have a complete description of the ideal N of R such that the Jacobson radical of $R[x]$ is equal to $N[x]$. As an easy consequence of the above we have

Corollary 2.2 *(KP) has a positive solution if and only if the Jacobson radical of a polynomial ring $R[x]$ is equal $Nil(R)[x]$.*

Another approach to study this is using primitive ideals. Recall that the Jacobson radical of a ring R is equal to the intersection of all the (right or left) primitive ideals of R. Thus the Jacobson radical of $R[x]$ could be determined if we can get a good description of $R/P \cap R$, for a primitive ideal P of $R[x]$. We easily have

Corollary 2.3 *The following conditions are equivalent:*

(i) *(KP) has a positive solution.*

(ii) *For any ring R and primitive ideal P of $R[x]$, $R/P \cap R$ does not have nil ideals.*

As an approximation to this question one can study the Brown-McCoy radical of a polynomial ring. A complete description of it was obtained in [19].

Recall that the Brown-McCoy radical of a ring R is defined as the intersection of all the ideals M of R such that R/M is a simple ring with an identity (when R itself has an identity this is clearly equal to the intersection of all the maximal ideals of R). We denote the Brown-McCoy radical of R by $U(R)$. We have $Nil(R) \subseteq J(R) \subseteq U(R)$, for any R. When R is a commutative ring, then $J(R) = U(R)$.

The Brown-McCoy radical of a polynomial ring in one indeterminate x can be described. First we have $U(R[X]) = I[X]$, where I is an ideal of R, for any set of (commuting or noncommuting) indeterminates X. This result was proved by Krempa in [14] for one indeterminate and in the general case in [17, 9].

The description given in [19] for the Brown-McCoy radical of $R[x]$ is as follows. Let $\mathcal{P}$ be the class of all prime rings with large center, i.e., rings such that for every nonzero ideal H of R, $H \cap Z(R) \neq 0$, where $Z(R)$ denotes the center of R. Then for any ring R the ideal I_1 such that $U(R[x]) = I_1[x]$ is equal to the intersection of all the ideals P of R such that $R/P \in \mathcal{P}$.

A ring R is said to be Brown-McCoy radical if $U(R) = R$. Equivalently, R cannot be homomorphically mapped onto a ring with an identity. As a consequence of the above description we have the following which is an approximation to (KP) (Theorem 2.1).

Theorem 2.4 [19] *If R is a nil ring, then $R[x]$ is Brown-McCoy radical.*

Theorem 2.4 was improved in [4]. In this interesting paper the authors proved the following stronger fact.

Theorem 2.5 *If R is a nil ring, then $R[x]$ cannot be homomorphically mapped onto a ring with a nonzero idempotent.*

In the attempt to solve (KP) several other related results have appeared into consideration. For example, Amitsur and Krempa [2, 13] asked whether for any nil ring R, $R[x]$ is nil. One can check that it is so for algebras over uncontable fields. Recently Smoktunowicz [24] constructed an example showing that the Amitsur-Krempa problem has a negative solution.

Another related conjecture by Amitsur is the following. If a polynomial ring in one indeterminate is Jacobson radical, then it is a nil ring. In [20] the authors gave an example showing that this conjecture is not true.

There are many other results and related problems in the literature on the subject. In particular, the reader may see the references included here and in the papers mentions in the introduction.

Now we present results on polynomial rings in several indeterminates which are proved in [6] and some open problems that are connected to these questions.

3 Polynomial Rings in Several Indeterminates

As we said in Section 2, the Brown-McCoy radical of a polynomial ring $R[X]$ in any set of (either commuting or noncommuting) inderminates X is equal to $I[X]$, for an ideal I of R, which clearly depends on R and X [14, 17, 9]. We denote this ideal by I_α when X is a set of commuting indeterminates of cardinality α. The description of I_1 have been obtained in [19] studying ideals M of $R[x]$ such that $R[x]/M$ is simple with identity.

The case of several indeterminates was studied in [6]. First we need to introduce the strongly prime radical of a ring.

A ring R is said to be (symmetric) strongly prime if R is prime and the central closure of R is simple ([26], Chap. 35). For rings with identities this notion and the related radical have been studied in [10]. Here we modify the definition for rings without identity element in such a way that the corresponding radical will be a radical very useful in our context.

Definition 3.1 *A ring R is said to be a unitary strongly prime (u-sp, for short) ring if R is prime and the central closure RC of R is simple with unit.*

Of course, if R itself has an identity, then this definition is equivalent to the one used in [10].

The notion defined above is left-right symmetric. We point out that it does not coincide with the more known notion of right (left) strongly prime rings [8].

An ideal P of a ring R is said to be u-sp if the factor ring R/P is a u-sp ring. The intersection of all the u-sp ideals of R is called the u-sp radical of R and will be denoted by $\mathcal{S}(R)$. The following was proved in [6].

Theorem 3.2 *Let R be a ring and X any set of either commuting or non-commuting indeterminates. Then $\mathcal{S}(R[X]) = \mathcal{S}(R)[X]$.*

It is easy to see that any prime ring with large center is u-sp. Then $\mathcal{S}(R) \subseteq I_1$, where I_1 is the ideal defined in the beginning of this section.

Let $\mathcal{P}$ denote the class of all prime rings with large center and $\mathcal{S}$ the class of all u-sp rings. We are interested in subclasses of $\mathcal{P}$ and $\mathcal{S}$. The pseudo radical $ps(R)$ of a ring R is defined as the intersection of all nonzero prime ideals of R. Then the class $\mathcal{P}_1$ (resp. $\mathcal{S}_1$) is defined as the class of all the rings in $\mathcal{P}$ (resp. $\mathcal{S}$) with nonzero pseudo radical. The following result was obtained in [6] and improves a result in [19].

Theorem 3.3 *For any ring R, the following conditions are equivalent:*

(i) *There exists an R-disjoint ideal M of* $R[x]$ *such that* $R[x]/M$ *is simple with identity.*

(ii) $R \in \mathcal{P}_1$.

(iii) *R is prime and* $ps(R) \cap Z(R) \neq 0$.

(iv) $R \in \mathcal{S}$ *and there exists* $c \in C$ *such that* $RC = R[c]$.

We extend the above to several indeterminates. Let X_n denote a set of n-commuting indeterminates. Then $R[X_n]$ means the usual polynomial ring $R[x_1, ..., x_n]$.

A finite subset $\{a_1, ..., a_m\} \subseteq R$ is called an insulator if for any $\{b_1, ..., b_s\}$ and $\{c_1, ..., c_s\}$ of elements of R we have that $\sum_i b_i a_j c_i = 0$, for all $j = 1, ..., m$, implies that $\sum_i b_i c_i = 0$. Note that if R is prime, when there exists a nonzero element $c \in ps(R) \cap Z(R)$, then $\{c\}$ is an insulator of R contained in $ps(R)$.

Theorem 3.4 *For any ring R, the following conditions are equivatent:*

(i) *There exists* $n \geq 1$ *and an R-disjoint ideal M of* $R[X_n]$ *such that* $R[X_n]/M$ *is simple with unit.*

(ii) $R \in \mathcal{S}_1$.

(iii) *R is prime and* $ps(R)$ *contains an insulator.*

(iv) $R \in \mathcal{S}$ *and for some m there exists* $c_1, ..., c_m$ *in C such that* $RC = R[c_1, ..., c_m]$

Theorem 3.4 allows us to obtain information on the Brown-McCoy radical of polynomial rings in several indeterminates.

It is easy to see that for any cardinal numbers $\alpha \geq \beta$ we have $I_\alpha \subseteq I_\beta$, where I_ρ denotes the ideal of R such that $U(R[X]) = I_\rho[X]$ and X a commuting set of ρ indeterminates. Also we can prove that $\mathcal{S}(R) \subseteq I_\rho$, for any cardinal ρ.

So we have a sequence

$$I_1 \supseteq I_2 \supseteq ... \supseteq I_\rho \supseteq \mathcal{S}(R),$$

for any ρ. We cannot give an answer to the following

Question 1 *Is there a ring R for which the above sequence is not constant?*

In an attempt to study this question we proved

Theorem 3.5 *Let R be a ring and X a set of either commuting or non-commuting indeterminates. If either X is infinite or R is a PI ring, then* $U(R[X]) = \mathcal{S}(R[X])(= \mathcal{S}(R)[X])$.

From Theorem 3.5 it follows that the sequence of Question 1 is constant when R is a PI ring and for any infinite cardinal ρ we have $I_\rho = \mathcal{S}(R)$.

Any prime PI ring has large center and is always u-sp. The question on whether a u-sp ring has always large center was raised by K. Beidar. It seems that this question is still open for u-sp rings with nonzero pseudo radical. Of course, a positive answer to this question would imply that Theorem 3.5 will be true for any ring and our Question 1 will have a negative answer. Moreover, in this case $I_n = \mathcal{S}(R)$, for any ring R and $n \geq 1$. To prove that this last relation holds it would be enough to give a positive answer to the following

Question 2 *Is it true that if $R \in \mathcal{S}_1$, then $ps(R)$ contains a nonzero central element?*

There is another question which is related to Question 1. The integers n and m in Theorem 3.4 and the number of elements of an insulator in $ps(R)$ are related. In fact, if R is prime and $ps(R)$ contains an insulator $\{a_1, ..., a_n\}$, then we can construct an ideal M of $R[X_m]$ such that $R[X_m]/M$ is simple with identity and $RC = R[c_1, ..., c_m]$, for some $c_1, ..., c_m \in C$, where $m \leq n$. However we do not know whether there exists an insulator with n elements in $ps(R)$ when there exists such an ideal M in $R[X_n]$.

Question 3 *Can the integers n and m in Theorem 3.4 assumed to be equal?*

We already know that if R is a nil ring, then $R[x]$ is Brown-McCoy radical. It is an open problem whether this is also true for two or more indeterminates [19]. On the other hand, it is also an open problem whether the nil radical of a ring is contained in the u-sp radical [10]. These two questions are connected:

Theorem 3.6 *The following conditions are equivalent:*

(i) *For any ring R, the nil radical is contained in the u-sp radical of R.*

(ii) *If R is a nil ring, then a polynomial ring over R in any finite number of commuting indeterminates is a Brown-McCoy radical ring.*

We finish the paper with the following

Question 4 *Are the equivalent conditions of Theorem 3.6 true?*

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