Resenhas IME-USP 2001, Vol. 5, No. 2, 107 - 111.

On the Sum of All N-commutators in a Finite Group[1]

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Abstract: In the theory of nonassociative algebras - like Lie algebras, alternative algebras, Jordan algebras - the Casimir operator is a useful tool in proving some important theorems. In my lecture I will discuss the Casimir operator in finite group algebras to give simple and elegant proofs of some known and unknown basic facts from group theory.

Key words: Finite group, Casimir element, character degree, commutator subgroup.

1 Background and notations

1.1

Let G be a finite group, $\rho_1, \ldots, \rho_r$ all irreducible representations of G with characters $\chi_i(g) = \operatorname{trace} \rho_i(g)$ and degrees $n_i = \chi_i(e)$. Let $\mathbb{C}G$ be the group algebra over the complex numbers and $Z(\mathbb{C}G)$ its center.
Then $r = \dim Z(\mathbb{C}G)$ = number of conjugacy classes of G. The class sums $C_1, \ldots, C_r$ form a basis of $Z(\mathbb{C}G)$, so do the central idempotents

$$e_i = \frac{n_i}{|G|} \sum_g \chi_i(g^{-1})g, \qquad e_i e_j = \delta_{ij} e_i \quad (1 \leq i, j \leq r) \tag{1}$$

1.2 Integrality

An element x in a unital ring R is called integral, if there exists a $k \in \mathbb{N}$ and integers a_i such that $x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0 e = 0$. Each $x = \sum_g \xi_g g \in \mathbb{C}G$ with integral coefficients $\xi_g \in \mathbb{Z}$ is integral. Thus, if we have a central element $\sum_g m_g g = \sum_i \alpha_i e_i$ with integral coefficients, then all α_i are algebraic integers, and therefore integers if $\alpha_i \in \mathbb{Q}$.

1.3 Casimir elements

Let A be a finite dimensional F-Algebra (F a field) with non degenerate, symmetric and associative bilinear form φ, and let
$(u_1, \ldots, u_n), (v_1, \ldots, v_n)$ be a pair of φ-dual bases, $\varphi(u_i, v_i) = \delta_{ij}$, then

$$c := \sum_{i=1}^{n} u_i v_i \tag{2}$$

[1]Short version of a talk given at the IX Encontro em Algebra USP-UNICAMP-UNESP, sept.2001. The author thanks the IME-USP, in particular I. Śhetakov, for their kind hospitality and FAPESP for support (Proc. 2001/02074-4).

is called a Casimir element of A. It is independent of the bases.

2 The Casimir element in $Z(\mathbb{C}G)$, G a finite group.

2.1

The bilinear form $\varphi(x,y) = \frac{1}{|G|}\sum_g \xi_g \eta_{g^{-1}}$ remains non degenerate when restricted to the center $Z(\mathbb{C}G)$. There we have 2 distinguished pairs of dual bases

$$\begin{array}{ll} (C_1,\ldots,C_r), & \left(\frac{|G|}{h_1}C_1',\ldots,\frac{|G|}{h_r}C_r'\right) \\ (e_1,\ldots,e_r), & \left(\left(\frac{|G|}{n_1}\right)^2 e_1,\ldots,\left(\frac{|G|}{n_r}\right)^2 e_r\right) \end{array}$$

Then, using (2) we get two different representations of the Casimir element which may already be used to prove a theorem of Zassenhaus on character degrees. But there is another amazing representation of c:

2.2 The main formula

Let X_g denote the set of all pairs $(a,b) \in G \times G$ such that $g = aba^{-1}b^{-1}$ and $f(g) = |X_g|$ its cardinality. Then

$$c = \sum_i \left(\frac{|G|}{n_i}\right)^2 e_i = \sum_{a,b\in G} aba^{-1}b^{-1} = \sum_g f(g)g. \tag{3}$$

Proof: We start with the decomposition of class sums as linear combination of the e_i

$$\sum_{a\in G} aba^{-1} = \sum_i \frac{|G|}{n_i}\chi_i(b)e_i,$$

multiply (from the right) by b^{-1}, sum over all b and use (1) to get the desired result. □

3 Applications

3.1 A generalization of a formula of Frobenius

Let $f_k(g)$ denote the number of $2k$-Tuples $(a_1,b_1,\ldots,a_k,b_k)$ such that $g = a_1b_1a_1^{-1}b_1^{-1}\ldots a_kb_ka_k^{-1}b_k^{-1}$, then

$$f_k(g) = \sum_i \left(\frac{|G|}{n_i}\right)^{2k-1} \chi_i(g) \tag{4}$$

Proof: Using (3) we compute c^k in 2 different ways

$$c_k = \sum_i \left(\frac{|G|}{n_i}\right)^{2k} e_i = \left(\sum_{a,b\in G} aba^{-1}b^{-1}\right)^k = \sum_g f_k(g)g;$$

replace e_i by its definition (1) and compare coefficients. (Frobenius formula is for $k-1$.) □

3.2 Elegant new proof

Elegant new proof of a basic theorem:

$$n_i \,|\, [G:Z] \qquad (Z \text{ center of } G).$$

Proof: $Z \times Z$ operates by leftmultiplication on X_g (see (2.2)): $(u,v)\cdot(a,b) = (ua, vb) \in X_g$ for $u, v \in Z, (a,b) \in X_g$. All orbits have equal length $|Z|^2$, thus $|Z|^2 | f(g)$. We then get from (3)

$$\sum_g \frac{f(g)}{|Z|^2} g = \sum_i \left(\frac{[G:Z]}{n_i}\right)^2 e_i,$$

thus $\frac{[G:Z]}{n_i} \in \mathbb{Z}$, by (1.2) □

3.3 A new result on G'

Theorem 1 *Any element from the commutator subgroup G' may be written as product of at most $s-1$ commutators, where s is the number of different character degrees.*

Proof: Using (3) - first equation - we find a relation $c^s = b_{s-1}c^{s-1} + \cdots + b_1 c + b_0 e$ with integral coefficients. And now - using the other representation of c in (3) - we see, that on the left hand side (in c^s) all products of s commutators occur, but each of them (right hand side) is already a product of $\leq s-1$ commutators. □

3.4 n-commutators.

The commutator subgroup G', by definition generated by all commutators, is the set of all n-commutators $a_1 a_2 \ldots a_n a_1^{-1} a_2^{-1} \ldots a_n^{-1}$, $n \geq 1$. The answer (which was not known before) to the following problems

1. determine the number $h_k(g)$ of n-tupels $(a_1, \ldots, a_n)$ such that $g = a_1 \ldots a_n a_1^{-1} a_2^{-1} \ldots a_n^{-1}$,
2. determine a lower bound for n,

will be given with the help of the following formula:

$$\sum_{a_1,\ldots,a_n \in G} a_1 \ldots a_n a_1^{-1} a_2^{-1} \ldots a_n^{-1} = \begin{cases} c^k, & \text{if } n = 2k \\ |G|c^k, & \text{if } n = 2k+1 \end{cases} \tag{5}$$

The short proof (by induction) uses $\sum_{a,b} abga^{-1}b^{-1} = gc$.
By the same method as used in (3.1) and (3.3) we easily derive from (5)

a) $h_{2k}(g) = f_k(g), \qquad h_{2k+1}(g) = |G| f_k(g)$

b) Each $g \in G'$ is a $2k$-commutator, $1 \leq k \leq s-1$.

3.5 Further results

M.Leitz used the Casimir element for an elementary proof of Reynold's Theorem:

$$\frac{|H|}{\theta(e)} | \frac{|G|}{\chi(e)}$$

where $H \leq G$, χ irreducible and θ an irreducible constituent of χ_H.
Tambour and Leitz gave (independent and different) proofs of Itô's theorem:

$$n_i | [G : A]$$

for abelian normal subgroups. Both (still elaborate) proofs use the sum of all commutators and/or the sum of all n-commutators.

A "nice" factorization $c_G = c_H \cdot w$ of the Casimir element c_G (resp. c_H for $H \leq G$) and w integral - which is not yet known - would simplify matters considerably.

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