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Domain Dependence of Elliptic Operators in Divergence Form[1]

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Abstract: We consider different metrics in the set Θ of all open sets, up to an equivalence relation, and analize the continuity under this metrics of the spectrum for second order elliptic differential operators in divergence form. In particular, we show that if $\{\Omega_k\}_{k\geq 0}$ is a sequence of domains in $B(0,1)$, if u^k are the solutions of $-\triangle u^k = 1$ with Dirichlet boundary conditions, and if $u^k \stackrel{k\to\infty}{\longrightarrow} u^0$ in $L^2(B(0,1))$, then for any elliptic operator $L = a_{ij}\partial_{ij} + b_i\partial_i + c$ with $a_{ij} \in C^{0,1}$, $b_i, c \in L^\infty$ the spectrum and spectral projections of L in Ω_k with Dirichlet boundary conditions converge to the spectrum and spectral projections of L in Ω_0 with Dirichlet boundary conditions.

Key words: Elliptic operators, divergence form, perturbation of domain, spectrum, spectral projections, metrics, space of open sets.

1 Introduction.

In [3], the author considered a general second order elliptic operator $L = a_{ij}\partial_{ij} + b_i\partial_i + c$, $a_{ij} \in C^0(I\!R^N)$, $b_i, c \in L^\infty(I\!R^N)$ and also the set $\Theta = \{\Omega \subset B(0,1), \text{open}\}$. He defined an equivalence relation $\sim$ in Θ and a metric d in $\tilde{\Theta} = \Theta/\sim$ and showed that the principal eigenvalue and eigenfunction of L, with Dirichlet boundary conditions, are continuous in this metric. The construction of the equivalence relation and the metric involve the solution of $Mu = -1$ where $M = a_{ij}\partial_{ij} + b_i\partial_i = L - c$ and therefore depend on the operator L. It will be desirable to construct a metric as simple as possible and independent of the operator L so that the same conclusions above still hold. In this work we will show that if we consider the class of operators L with $a_{ij} \in C^{0,1}$ then this is possible. The main idea is to use the solutions of $\triangle u = -1$ in Ω to define the metric. Since $a_{ij} \in C^{0,1}$ we can use variational techniques to, roughly speaking, compare both operators, $\triangle$ and L.

To fix the ideas let us recall the construction from [3].

The equivalence relation in Θ is constructed as follows. Let $L = a_{ij}\partial_{ij} + b_i\partial_i + c$ with $a_{ij} \in C^0(I\!R^N)$, $b_i, c \in L^\infty(I\!R^N)$ an strictly elliptic operator and define $M = a_{ij}\partial_{ij} + b_i\partial_i = L - c$. Let Ω be a bounded open set $\Omega \subset B(0,1)$. Following the ideas of [7], we first define a canonical function $u^{\Omega,M}$ depending only on Ω and on the operator M. Consider a family of smooth domains H_j, $j \geq 1$, so that $\bar{H}_j \subset \Omega$, $\bar{H}_j \subset H_{j+1}$ and $\bigcup_{j\geq 1} H_j = \Omega$. If $M = a_{ij}\partial_{ij} + b_i\partial_i$, so that $L = M + c$, denote by u_j the solution of,

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$$\begin{aligned} Mu_j = & \ -1, \qquad H_j, \\ u_j = & \ \ 0, \qquad \partial H_j. \end{aligned} \tag{1.1}$$

This solution exists, since H_j is smooth, and it is positive in H_j by the maximum principle. It can be proved that u_j is a nondecreasing sequence which converges to a function $u^{\Omega,M}$ weakly in $W^{2,p}(J)$, for any $p > 1$, and strongly in $C^1(J)$, for any compact set $J \subset \Omega$. Moreover, $u^{\Omega,M}$ is a strong solution of the problem $Mu = -1$ in Ω. It is not difficult to prove that the function $u^{\Omega,M}$ is independent of the domains H_j and only depends on Ω and the operator M. Moreover $u^{\Omega,M} > 0$ in Ω. In general it is not true that $u^{\Omega,M}(x) \overset{x \to \partial\Omega}{\longrightarrow} 0$, although if $z \in \partial\Omega$ is a regular point of the boundary, in the sense that exists a barrier at that point, we have $u^{\Omega,M}(x) \overset{x \to z}{\longrightarrow} = 0$, (see [7]). This function was of fundamental importance in the analysis of [3] and [7].

Defining $\Gamma_{\Omega,M} = \{x \in \partial\Omega : \exists\{x_k\} \subset \Omega,\ x_j \to x \text{ and } u^{\Omega,M}(x_j) \to 0\}$, we can construct the open set $\Omega^{*,M} = \bar{\Omega} \setminus \Gamma_{\Omega,M}$ (see [3]). The equivalence relation is defined as follows

$$\Omega_1 \sim_M \Omega_2 \Longleftrightarrow \Omega_1^{*,M} = \Omega_2^{*,M} \tag{1.2}$$

With this equivalence relation we define $\tilde{\Theta}^M = \Theta / \sim_M$

The metric constructed in [3] is the following

$$\begin{aligned} d^M_{L^\infty} : \tilde{\Theta}^M \times \tilde{\Theta}^M \ & \longrightarrow I\!R^+ \\ (\Omega_1, \Omega_2) \ & \longrightarrow d^M_{L^\infty}(\Omega_1, \Omega_2) \equiv \|u^{\Omega_1,M} - u^{\Omega_2,M}\|_{L^\infty(B_1)} \end{aligned} \tag{1.3}$$

It is showed in [3] that $(\tilde{\Theta}^M, d^M_{L^\infty})$ is a complete metric space. Moreover, the principal eigenvalue and eigenfunction of L are continuous in this metric. Similar continuity results are also obtained for the solution of the equation $Lv = f$.

At this point several important questions arise:

(1). How does the equivalence relation $\sim_M$ depend on the operator M? Is it possible to show that the equivalence relation is independent of M, at least for certain class of operators M?

(2). Is it possible to obtain similar results for metrics of the type $d^M_{L^p}(\Omega_1, \Omega_2) = \|u^{\Omega_1,M} - u^{\Omega_2,M}\|_{L^p(B_1)}$ for certain $1 \leq p < \infty$ or $d^M_{H^1}(\Omega_1, \Omega_2) = \|u^{\Omega_1,M} - u^{\Omega_2,M}\|_{H^1(B_1)}$?

(3). How do all these metrics depend on the operator M? Is it possible to show that $d^M_{L^p}(\Omega_k, \Omega_0) \overset{k\to\infty}{\longrightarrow} 0$ if and only if $d^{\tilde{M}}_{L^p}(\Omega_k, \Omega_0) \overset{k\to\infty}{\longrightarrow} 0$ for M, $\tilde{M}$ two different operators, maybe in certain class?

(4). Is it possible to obtain some continuity in these metrics for the rest of the spectrum?

In this work we will treat these questions for the class of operators L with $a_{ij} \in C^{0,1}$ and that, therefore, can be expressed in divergence form.

Let us define the following families of operators:

$$\mathcal{D} = \{L = \partial_i(a_{ij}\partial_j) + b_i\partial_i + c;\ a_{ij} \in C^{0,1}(\mathbb{R}^n),\ b_i, c \in L^\infty(\mathbb{R}^n),\ \text{and such that there exists } \nu > 0, \text{ with } a_{ij}\xi_i\xi_j \geq \nu|\xi|^2\}$$

$$\mathcal{D}_0 = \{L \in \mathcal{D};\ c \equiv 0\}$$

$$\mathcal{S} = \{M \in \mathcal{D};\ b_i \equiv 0\}$$

$$\mathcal{S}_0 = \{M \in \mathcal{D};\ b_i \equiv 0,\ c \equiv 0\}$$

Notice that if $L \in \mathcal{D}$, it can also be written as $L = a_{ij}\partial_{ij} + \tilde{b}_i\partial_i + c$ with $\tilde{b}_i = \partial_j(a_{ij}) + b_i$.

In this paper all the operators will be considered with Dirichlet boundary conditions. Therefore, notions like: spectrum, resolvent operator, selfadjointness, etc. are related only to Dirichlet boundary conditions. Also, B_1 will always represent the unit ball $B(0,1)$ in $\mathbb{R}^N$. All the open sets considered in this work will be bounded, and when a sequence of open sets $\{\Omega_k\}$ is considered it will always be uniformly bounded. Hence, without loss of generality we can assume that $\Omega_k \subset B_1$.

In Section 3, we will show that for any $M, \tilde{M} \in \mathcal{D}_0$ and any $\Omega \subset B_1 \equiv B(0,1)$, $\Omega^{*,M} = \Omega^{*,\tilde{M}}$. Therefore the equivalence relation is the same for any operator $M \in \mathcal{D}_0$. This allows us to define $\Omega^* \equiv \Omega^{*,M}$ and $\tilde{\Theta} \equiv \tilde{\Theta}^M$, for any $M \in \mathcal{D}_0$.

With respect to the metric, we will see in Section 5 that for a fixed $M \in \mathcal{D}_0$ the metrics $d^M_{L^p}$, $1 \leq p < \infty$ and $d^M_{H^1}$ define the same metric. Moreover if $M \in \mathcal{S}_0$, $\tilde{M} \in \mathcal{D}_0$ we have,

$$d^M_{L^p}(\Omega_k, \Omega_0) \overset{k\to\infty}{\longrightarrow} 0 \Longrightarrow d^{\tilde{M}}_{L^p}(\Omega_k, \Omega_0) \overset{k\to\infty}{\longrightarrow} 0.$$

In particular, if $M, \tilde{M} \in \mathcal{S}_0$ then the metrics $d^M_{L^p}$ and $d^{\tilde{M}}_{L^p}$ are equivalents.

In Section 4, we will see that if $L \in \mathcal{D}$ and $M = L - c$ then the spectrum, spectral projections and the resolvent operator of L, behave continuously in the metric $d^M_{L^p}$.

Notice that the metric $d^M_{L^p}$ is strictly weaker than the metric $d^M_{L^\infty}$, defined in [3]. For example, if Ω is a domain, $p \in \Omega$ and we define $\Omega_k = \Omega \setminus B(p, \frac{1}{k})$ we will have $d^M_{L^p}(\Omega_k, \Omega) \overset{k\to\infty}{\longrightarrow} 0$ while $d^M_{L^\infty}(\Omega_k, \Omega) \geq \delta > 0$. This means that with the metrics $d^M_{L^p}$ we may allow more general perturbations of the domain. Moreover, while the space $(\tilde{\Theta}, d^M_{L^\infty})$ is a complete metric space we will show that the space $(\tilde{\Theta}, d^M_{L^p})$, $1 \leq p < \infty$ is not complete, see Section 6.

Since the Laplace operator, $\triangle$, is the simplest second order elliptic differential operator it is natural to define a canonical metric, d_2, by

$$d_2(\Omega_1, \Omega_2) = \|u^{\Omega_1,\triangle} - u^{\Omega_2,\triangle}\|_{L^2}$$

With the results of this paper the following convergence result can be proved

Theorem 1.1 *Let $\{\Omega_k\}_{k\geq 0} \subset \Theta$ and let u^k be the solution of $\triangle u^k = -1$ in Ω_k, $u^k = 0$ in $\partial\Omega_k$ for any $k \geq 0$. Let $L \in \mathcal{D}$ and denote by L_k the operator L with Dirichlet boundary conditions acting in Ω_k*

Consider the following statements:

(i). $u^k \overset{k\to\infty}{\longrightarrow} u^0$ in $L^2(B_1)$ (that is $d_2(\Omega_k, \Omega_0) \overset{k\to\infty}{\longrightarrow} 0$).

(ii). The spectrum of L_k approaches the spectrum of L_0, and the spectral projections of L_k approach the spectral projections of L_0 in $\mathcal{L}(L^2(B_1), H_0^1(B_1))$.

Then (i) implies (ii).

Moreover, if L is selfadjoint, both statements are equivalent.

This theorem is proved in Section 5.

In [20], [21], [22], [23], Micheletti studies the properties of the spectrum of an elliptic operator when the domain is perturbed. She constructs and studies a metric in the family of bounded smooth domains, Courant's metric, and obtains generic properties of the eigenvalues in this metric. The domains are smooth, and the metric is stronger than the metric d_2, constructed in this paper. Therefore we are able to allow more general perturbations than the ones in [20]-[23].

Most of the results in the literature related to the behavior of the spectrum of an operator when the domain is perturbed (regula⁻ or singularly) put the emphasis on geometric conditions on the perturbation of the domain to garantee the continuity of the spectrum. In this sense our work is different. The conditions we imposse are not geometric: we reduce the continuity of the spectrum of a second order elliptic operator in divergence form, to obtain the continuity in L^2 of the solutions of one of the simplest nontrivial elliptic equations, $-\triangle u = 1$, in the perturbed domains.

For other results related to the different behavior of the operator L when the domain is perturbed (continuity results or singular behavior) under different kind of boundary condition the reader is referred to the bibliography at the end of the article.

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2 Preliminaries

In this section we prove some preliminary results that will be used throught the paper.

Lemma 2.1 *Let $a_{ij} \in L^\infty(\Omega)$ such that there exists a $\nu > 0$ with $a_{ij}\xi_i\xi_j \geq \nu|\xi|^2$. Assume $\phi_k \in H^1(\Omega)$, $\phi_k \overset{k\to\infty}{\longrightarrow} \phi$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Then, if $\int_\Omega a_{ij}\partial_i\phi_k\partial_j\phi_k \overset{k\to\infty}{\longrightarrow} \int_\Omega a_{ij}\partial_i\phi\partial_j\phi$, we have $\phi_k \overset{k\to\infty}{\longrightarrow} \phi$ strongly in $H^1(\Omega)$.*

Proof. If we define $\|f\|_a = \int_\Omega a_{ij}\partial_i f \partial_j f + \int_\Omega |f|^2$, then $\|\cdot\|_a$ is a norm in $H^1(\Omega)$ equivalent to the usual norm. Moreover, for the sequence ϕ_k we have $\|\phi_k\|_a \stackrel{k\to\infty}{\longrightarrow} \|\phi\|_a$. This fact and the weak convergence in $H^1(\Omega)$ of ϕ_k imply the strong convergence in $H^1(\Omega)$.

Lemma 2.2 *Let $L = \partial_i(a_{ij}\partial_j) + b_i\partial_i + c \in \mathcal{D}$ with ellipticity constant ν and such that $\|a_{ij}\|_{L^\infty}, \|b_i\|_{L^\infty}, \|c\|_{L^\infty} \le \beta$ for certain positive constant β. Define also $M = \partial_i(a_{ij}\partial_j) + b_i\partial_i = L - c \in \mathcal{D}_0$. We have,*
(i). There exist constants η and c_0 depending only on ν and β, such that for any open set $\Omega \subset B_1 \subset \mathbb{R}^N$, the operator $-L+\eta$ defines a bilinear coercive form $l_\eta(u,v)$, in $H_0^1(\Omega)$ with $|l_\eta(u,v)| \le c_0\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}$ and $l_\eta(u,u) \ge c_0\|u\|^2_{H_0^1(\Omega)}$.
(ii). Let $p \in (\frac{2N}{N+2}, 2]$. There exists a constant c_1, depending on ν, β and p, such that for any $\Omega \subset B_1$ and for any $f \in L^p(\Omega)$ there exists a unique $w \in H_0^1(\Omega)$ solution of $Lw - \eta w = f$ and $\|w\|_{H_0^1(\Omega)} \le c_1\|f\|_{L^p(\Omega)}$. In particular, the resolvent operator $R(\lambda, L)$ is compact.
(iii). There exist a constant c_2, depending only on ν and β such that if $\Omega \subset B_1$, $f \in L^\infty(\Omega)$ and $Mv = f$ then $\|v\|_{H^1(\Omega)} \le c_2\|f\|_{L^\infty(\Omega)}$

Proof. (i). By definition

$$l_0(u,v) = \int_\Omega a_{ij}\partial_i u \partial_j v - \int_\Omega b_i \partial_i u v - \int_\Omega cuv$$

Notice that $|\int_\Omega b_i\partial_i uv| \le \|b\|_\infty \|u\|_{H_0^1}\|v\|_{L^2} \le \|b\|_\infty(\epsilon\|u\|^2_{H_0^1} + \frac{1}{\epsilon}\|v\|^2_{L^2})$
Similarly $|\int_\Omega cuv| \le \|c\|_\infty(\epsilon\|u\|_{L^2} + \frac{1}{\epsilon}\|v\|_{L^2})$
This implies that

$$l_\eta(u,u) \ge (\alpha - \|b\|_\infty\epsilon)\|\nabla u\|^2_{L^2(\Omega)} + (\eta - \frac{\|b\|_\infty + \|c\|_\infty}{\epsilon})\|u\|_{L^2}$$

Choosing $\epsilon = \frac{\alpha}{2\|b\|_\infty}$ and $\eta = \frac{\|b\|_\infty + \|c\|_\infty}{\epsilon} + \frac{\alpha}{2}$ we obtain $l_\eta(u,u) \ge \frac{\alpha}{2}\|u\|^2_{H_0^1(\Omega)}$.

(ii). If $p \in (\frac{2N}{N+2}, 2]$ and $1/p + 1/q = 1$, we have $H_0^1(B_1) \hookrightarrow L^q(B_1)$, with embedding constant C. Therefore, $|\int_\Omega fv| \le \|f\|_{L^p(\Omega)} \cdot \|v\|_{L^q(\Omega)} \le C\|f\|_{L^p(\Omega)} \cdot \|v\|_{H_0^1(\Omega)}$. This means that if $f \in L^p(\Omega)$ then f defines a continuous linear functional F in $H_0^1(\Omega)$ by $F(v) = \int_\Omega fv$ and

$$\|F\| = \sup\{\int_\Omega fv : \|v\|_{H_0^1(\Omega)} = 1\} \le C\|f\|_{L^p(\Omega)}$$

By the Lax-Milgram theorem we get the existence of a unique $w \in H_0^1(\Omega)$ satisfying $l_\eta(w,v) = F(v)$ for all $v \in H_0^1(\Omega)$ and $\|u\|_{H_0^1(\Omega)} \le \frac{2C}{\alpha}\|f\|_{L^p(\Omega)}$.

(iii). From the maximum principle we know that $\|v\|_{L^\infty(\Omega)} \le B\|f\|_{L^\infty(\Omega)}$ with B independent of Ω. Choosing the constant η of (i), we get that $Mv - \eta v = f - \eta v$ and therefore from (ii) $\|v\|_{H_0^1(\Omega)} \le c_1\|f - \eta v\|_{L^2(\Omega)} \le c_1|\Omega|^{\frac{1}{2}}(1 + |\eta|B)\|f\|_{L^\infty(\Omega)}$.

This proves the lemma.

Consider also the following result

Lemma 2.3 *Let $M \in \mathcal{D}_0$. The functions u_j used in the definition of $u^{\Omega,M}$, (statement (1.1)), satisfy $u_j \stackrel{j\to\infty}{\longrightarrow} u^{\Omega,M}$ in $H_0^1(\Omega)$. Hence, $u^{\Omega,M} \in H_0^1(\Omega)$ and therefore it is the unique weak solution of $Mu = -1$ in Ω, $u = 0$ in $\partial\Omega$.*

Proof. From Lemma 2.2, we have uniform bounds on $\|u_j\|_{H_0^1(\Omega)}$. This implies that, by getting a subsequence, there exists a function $v \in H_0^1(\Omega)$ and $u_j \stackrel{j\to\infty}{\longrightarrow} v$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Since we already know that $u_j \stackrel{j\to\infty}{\longrightarrow} u^{\Omega,M}$ in $L^\infty_{loc}(\Omega)$ then $v = u^{\Omega,M}$. The strong convergence in $H_0^1(\Omega)$ follows from Lemma 2.1 and the fact that $\int_\Omega a_{rs}\partial_r u_j \partial_s u_j = \int_\Omega (b_r \partial_r u_j u_j + u_j) \stackrel{j\to\infty}{\longrightarrow} \int_\Omega (b_r \partial_r u^{\Omega,M} u^{\Omega,M} + u^{\Omega,M}) = \int_\Omega a_{rs}\partial_r u^{\Omega,M}\partial_s u^{\Omega,M}$.

3 On the Equivalence Relation.

In this section we show that the equivalence relation $\sim_M$, defined by (1.2), is independent of the operator $M \in \mathcal{D}_0$.

Let us start by proving the following,

Lemma 3.1 *If $\Omega \in \Theta$ and $M \in \mathcal{D}_0$, then $H_0^1(\Omega) = H_0^1(\Omega^{*,M})$.*

Proof. To simplify notation, denote by $\Omega^* = \Omega^{*,M}$, $u = u^{\Omega,M}$, $u^* = u^{\Omega^*,M}$. Since $\Omega \subset \Omega^*$ we have $H_0^1(\Omega) \subset H_0^1(\Omega^*)$. Therefore, we just need to show that if $\phi \in C_0^\infty(\Omega^*)$ then $\phi \in H_0^1(\Omega)$.

From [3], Theorem 2.12 (ii), we have that $u = u^*$ a.e.. Hence $u^* \in H_0^1(\Omega_0)$. Let u_j be the functions defined in (1.1). From Lemma 2.3, $u_j \stackrel{j\to\infty}{\longrightarrow} u = u^*$ in $H_0^1(\Omega)$. Consider the function $\phi_j = \phi\frac{u_j}{u^*} \in H_0^1(\Omega)$. Notice that since ϕ has compact support in Ω^* then $u^* \geq \alpha > 0$ in the support of ϕ and also $\phi\frac{u_j}{u^*} \stackrel{j\to\infty}{\longrightarrow} \phi$ in $H_0^1(\Omega)$. This implies that $\phi \in H_0^1(\Omega)$. This proves the lemma.

Lemma 3.2 *Let $x \in \partial\Omega$. Let $M, \tilde{M} \in \mathcal{D}_0$. There exist α, $r > 0$ such that $u^{\Omega,M}(y) > \alpha$ for $y \in B(x,r) \cap \Omega$ if and only if there exists $\tilde{\alpha}$, $\tilde{r} > 0$ such that $u^{\Omega,\tilde{M}}(y) > \tilde{\alpha}$ for $y \in B(x,\tilde{r}) \cap \Omega$*

Proof. Let us assume that there exists α, $r > 0$ such that $u^{\Omega,M}(y) > \alpha$ for $y \in B(x,r) \cap \Omega$. Notice that from the definition of $\Omega^{*,M}$ we have that $B(x,r) \subset \Omega^{*,M}$ (see also [3]). From the previous lemma, this implies that $H_0^1(\Omega) = H_0^1(\Omega \cup B(x,r)) = H_0^1(\Omega^{*,M})$.

If $u^{\Omega,\tilde{M}}$ is the unique $H_0^1(\Omega)$ solution of $\tilde{M} = -1$. ¿From the previous observation we get that $u^{\Omega,\tilde{M}} \in H_0^1(\Omega \cup B(x,r))$, and it is the unique $H_0^1(\Omega \cup B(x,r))$ solution of $\tilde{M}u = -1$. From this and the maximum principle, it is clear that there exists an $\tilde{\alpha} > 0$ such that $u^{\Omega,\tilde{M}} > \tilde{\alpha}$ in $B(x,r/2) \cap \Omega$. This proves the lemma.

With this lemma it is easy to prove the following result

Proposition 3.3 *The equivalence relation defined in the set Θ is independent of the operator $M \in \mathcal{D}_0$. That is $\Omega^{*,M} = \Omega^{*,\tilde{M}}$ for any M, $\tilde{M} \in \mathcal{D}_0$.*

Proof. The proof is simple. From the previous lemma we have that $\{x \in \partial\Omega : \exists\, \alpha, r > 0, u^{\Omega,M}(y) > \alpha, \forall y \in B(x,r) \cap \Omega\} = \{x \in \partial\Omega : \exists\, \alpha, r > 0, u^{\Omega,\tilde{M}}(y) > \alpha, \forall y \in B(x,r) \cap \Omega\}$. This implies that $\partial\Omega \setminus \Gamma_{\Omega,M} = \partial\Omega \setminus \Gamma_{\Omega,\tilde{M}}$. In particular $\Gamma_{\Omega,M} = \Gamma_{\Omega,\tilde{M}}$. This proves the proposition.

Remark 3.4 *Since $\Omega^{*,M}$ is independent of $M \in \mathcal{D}_0$, we can define $\Omega^* \equiv \Omega^{*,M}$, for any $M \in \mathcal{D}_0$, and the set of all equivalence class $\tilde{\Theta} \equiv \tilde{\Theta}^M$. Also, in view of the results of [3], the properties of the operator $L \in \mathcal{D}$ in Ω and Ω^* are exactly the same. Therefore, from now on, when an open set Ω is considered we can always assume that $\Omega = \Omega^*$. We will considered this facts without further mentioning to it.*

4 On the Continuity of the whole spectrum.

Let us start with the following result:

Lemma 4.1 *Let $\{\Omega_k\}_{k\geq 0}$ be a sequence of open sets with $\Omega_k \subset B_1$, $M \in \mathcal{D}_0$ and let $u^k \equiv u^{\Omega_k,M}$. The following statements are equivalent,*
(i). $u^k \stackrel{k\to\infty}{\longrightarrow} u_0$ in $L^p(B_1)$ for some $p \in [1,\infty)$,
(ii). $u^k \stackrel{k\to\infty}{\longrightarrow} u_0$ in $L^p(B_1)$ for all $p \in [1,\infty)$,
(iii). $u^k \stackrel{k\to\infty}{\longrightarrow} u_0$ in $H^1(B_1)$.

Proof. We know from the maximum principle that the functions u^k are uniformly bounded in L^∞. From this, it is clear that $(iii) \Rightarrow (i) \Leftrightarrow (ii)$. Let us see that $(ii) \Rightarrow (iii)$. Assume $u^k \stackrel{k\to\infty}{\longrightarrow} u^0$ in $L^p(B_1)$ for all $p \in [1,\infty)$. From Lemma 2.2 (iii), we have that there exists a constant C independent of k, such that $\|u^k\|_{H^1_0(B)} \leq C$. Therefore, we can get a subsequence $u^{k'}$ and a function $v \in H^1_0(B)$ such that $u^{k'} \longrightarrow v$ weakly in $H^1_0(B)$ and strongly in $L^2(B)$. Since $u^k \stackrel{k\to\infty}{\longrightarrow} u^0$, we have that $v = u^0$ and that the whole sequence u^k converges to u^0 weakly in $H^1_0(B)$. Moreover,

$$\begin{aligned}\int_B a_{ij}\partial_i u^k \partial_j u^k = \quad & \int_{\Omega_k} a_{ij}\partial_i u^k \partial_j u^k = \int_{\Omega_k} u^k + \int_{\Omega_k} b_i \partial_i (u^k) u^k \stackrel{k\to\infty}{\longrightarrow} \\ & \int_{\Omega_0} u^0 + \int_{\Omega_0} b_i \partial_i (u^0) u^0 = \int_{\Omega_0} a_{ij}\partial_i u^0 \partial_j u^0 = \int_B a_{ij}\partial_i u^0 \partial_j u^0\end{aligned}$$

This last statement and Lemma 2.1 prove the lemma.

As usual we denote by $\sigma(T)$ and by $\rho(T)$ the spectrum and the resolvent set of the operator T.

The operator $L \in \mathcal{D}$ with Dirichlet boundary conditions in Ω has a resolvent operator $R(\lambda,\Omega) : L^2(\Omega) \longrightarrow L^2(\Omega)$. Notice that we can also consider this operator from $L^2(B_1) \longrightarrow L^2(B_1)$. If $\phi \in L^2(B_1)$ we define $R(\lambda,\Omega)\phi = R(\lambda,\Omega)(\phi_{|\Omega}) \in H_0^1(\Omega) \hookrightarrow H_0^1(B_1) \hookrightarrow L^2(B_1)$. Also, it is easy to see that the non-zero spectrum of $R(\lambda,\Omega)$ in $L^2(\Omega)$ and in $L^2(B_1)$ coincide.

Since the spectral projections are given by $P = \frac{1}{2\pi i}\int_\Gamma R(\lambda,\Omega)d\lambda$ for certain Jordan curves Γ in the complex plane, see [17], we can also consider this projections from $L^2(B_1) \longrightarrow H_0^1(B_1) \hookrightarrow L^2(B_1)$.

With respect to the behavior of the spectrum and the resolvent operators, we have the following result.

Proposition 4.2 *Let $\{\Omega_k\}_{k\geq 0}$ be a sequence of open sets with $\Omega_k \subset B_1$. Let $L = \partial_i(a_{ij}\partial_j) + b_i\partial_i + c \in \mathcal{D}$ and denote by $M = \partial_i(a_{ij}\partial_j) + b_i\partial_i \in \mathcal{D}_0$. Denote by L_k the operator L with Dirichlet boundary conditions acting on Ω_k. If $u^{\Omega_k,M} \overset{k\to\infty}{\longrightarrow} u^{\Omega_0,M}$ in $L^2(B_1)$, that is $d_2^M(\Omega_k,\Omega_0) \overset{k\to\infty}{\longrightarrow} 0$, then the following statements are true:*
(i). For any C^1-Jordan curve Γ in the complex plane such that $\Gamma \cap \sigma(L_0) = \emptyset$, there exists a $k_0 = k(\Gamma)$ such that $\Gamma \cap \sigma(L_k) = \emptyset$ for $k \geq k_0$. Moreover, if P_{Γ,L_k} are the spectral projection over the part of the spectrum inside Γ, we have $\|P_{\Gamma,L_k} - P_{\Gamma,L_0}\|_{\mathcal{L}(L^2(B_1),H_0^1(B_1))} \overset{k\to\infty}{\longrightarrow} 0$
(ii). If $R(\lambda,L_k)$ is the resolvent operator of L_k, then

$$\|R(\lambda,L_k) - R(\lambda,L_0)\|_{\mathcal{L}(L^2(B_1),H_0^1(B_1))} \overset{k\to\infty}{\longrightarrow} 0,$$

and this convergence is uniform in a compact set $\Gamma \subset \rho(L_0)$.

Proof. Let us choose the constant η given by Lemma 2.2 so that the conclusions of the lemma are valid for the operators $L_k - \eta$. In particular $\eta \in \rho(L_k)$ for all k. Let us prove that $\|R(\eta,L_k) - R(\eta,L_0)\|_{\mathcal{L}(L^2(B_1),H_0^1(B_1))} \overset{k\to\infty}{\longrightarrow} 0$. This is equivalent to show that if $f_k \in L^2(B_1)$ and $f_k \overset{k\to\infty}{\longrightarrow} f_0$ weakly in $L^2(B_1)$ and if $v_k \in H_0^1(\Omega_k)$ are the solutions of $L_kv_k - \eta v_k = f_k$ for all $k \geq 0$, then we can get a subsequence, denoted by v_k again, such that $v_k \overset{k\to\infty}{\longrightarrow} v_0$ strongly in $H_0^1(B_1)$.

Denote by $T_k = L_k - \eta$. We know that there exists a constant c_1 independent of k such that $\|v_k\|_{H_0^1(B_1)} \leq C\|f_k\|_{L^2(B_1)}$. Therefore we can get a subsequence of $\{v_k\}$ and a function $v \in H_0^1(B_1)$, such that $v_k \overset{k\to\infty}{\longrightarrow} v$ weakly in $H_0^1(B_1)$ and strongly in $L^2(B_1)$.

Let $\phi \in C_0^\infty(\Omega_0)$. Since $\mathrm{supp}(\phi) \subset\subset \Omega_0$, we have that there exists an $\alpha > 0$ such that $u^0 > \alpha$ in $\mathrm{supp}(\phi)$. Define the function $\phi_k = \phi \cdot \frac{u^k}{u^0} \in H_0^1(\Omega_0) \cap H_0^1(\Omega_k)$, for k large enough, and satisfies $\phi_k \overset{k\to\infty}{\longrightarrow} \phi$ in $H_0^1(\Omega_0)$. We have,

$$-\int_{\Omega_0} a_{ij}\partial_i(v_k)\partial_j(\phi_k) + \int_{\Omega_0} b_i\partial_i(v_k)\phi_k + \int_{\Omega_0} cv_k\phi_k - \int_{\Omega_0} \eta v_k\phi_k = \int_{\Omega_0} f_k\phi_k$$

Taking the limit as $k \to \infty$ in the above expression, we get

$$-\int_{\Omega_0} a_{ij}\partial_i(v)\partial_j(\phi) + \int_{\Omega_0} b_i\partial_i(v)\phi + \int_{\Omega_0} cv\phi - \int_{\Omega_0} \eta v\phi = \int_{\Omega_0} f_0\phi$$

We claim that $v \in H_0^1(\Omega_0)$.

If we assume the claim proved then $v = v_0$. Moreover, from the above expressions it is easy to see that $\int_B a_{ij}\partial v_k \partial_j v_k \stackrel{k\to\infty}{\longrightarrow} \int_B a_{ij}\partial v_0 \partial_j v_0$, which implies, from Lemma 2.1, that $v_k \to v_0$ strongly in $H_0^1(B)$.

Let us prove the claim.

Let $M \in Z^+$ and denote by $f_k^M(x) = \min\{M, \max\{f_k(x), -M\}\}$ the cut-off function of f_k at a height M. Let v_k^M the solution of $T_k v_k^M = f_k^M$. By the maximum principle we have $|v_k^M(x)| \le u^k(x)M$, and from Lemma 2.2, we have that $\|v_k^M\|_{H_0^1(B)} \le c_1 M|\Omega|$. With a diagonalization procedure we can get a subsequence, denoted again with the index k, and functions v^M such that $v_k^M \stackrel{k\to\infty}{\longrightarrow} v^M$ weakly in $H_0^1(B)$. Since $|v_k^M(x)| \le u^k(x)M$, we have $|v^M(x)| \le u^0(x)M$ and therefore $v^M \in H_0^1(\Omega_0)$.

Choose $p \in (\frac{2n}{N+2}, 2)$. From Lemma2.2 we know that $\|v_k^M - v_k\|_{H_0^1(B)} \le c_1\|f_k^M - f_k\|_{L^p(B)}$.

But

$$\|f_k^M - f_k\|_{L^p(B)}^p = \int_{\{x:\,|f_k(x)|>M\}} |f_k|^p \le \|f_k\|_{L^2(B)}^p \cdot |\{x : |f_k(x)| > M\}|^{(2-p)/2}$$

Since $f_k \stackrel{k\to\infty}{\longrightarrow} f_0$ then $\|f_k\|_{L^2} \le C$ independent of k and therefore there exists a function $g(M) \to 0$ as $M \to \infty$ such that $|\{x : |f_k(x)| > M\}| \le g(M)$. Hence, $\|v_k^M - v_k\|_{H_0^1(B)} \le c_1 C[g(M)]^{(2-p)/2}$. Letting $k \to \infty$ we obtain $\|v^M - v\|_{H_0^1(B)} \le c_1 C[g(M)]^{(2-p)/2}$. From the fact that $v^M \in H_0^1(\Omega_0)$ and letting $M \to \infty$ in this last expression we obtain that $v \in H_0^1(\Omega_0)$.

This shows that $\|R(\eta, L_k) - R(\eta, L_0)\|_{\mathcal{L}(L^2(B_1), H_0^1(B_1))} \stackrel{k\to\infty}{\longrightarrow} 0$.

To simplify the notation let us denote by

$$R_\lambda^k = R(\lambda, L_k), \qquad \|\cdot\|_2 = \|\cdot\|_{\mathcal{L}(L^2(B_1), L^2(B_1))}$$

$$\|\cdot\|_1 = \|\cdot\|_{\mathcal{L}(H_0^1(B_1), H_0^1(B_1))} \quad \text{and } \|\cdot\|_{2,1} = \|\cdot\|_{\mathcal{L}(L^2(B_1), H_0^1(B_1))}$$

In the notation of Kato, [17], the operators R_η^k converge to R_η^0 in the *generalized sense*, (see Section IV.2.6 of [17]). This implies that $\sigma(R_\eta^k)$ converges to $\sigma(R_\eta^0)$ (see Section IV.3.5 of [17]). Hence, since $\lambda \in \sigma(L_k)$ if and only if $\frac{1}{\eta-\lambda} \in \sigma(R_\eta^k)$, we also have the convergence of $\sigma(L_k)$ to $\sigma(L_0)$.

It is now an standard procedure to prove (ii). If $\Gamma \subset\subset \rho(L_0)$ then there exists a k_0 such that $\Gamma \subset\subset \rho(L_k)$ for all $k \ge k_0$. If $\lambda \in \Gamma$, using the resolvent equation $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu$ it is easy to prove that,

$$R_\lambda^k - R_\lambda^0 = [I - (\lambda - \eta)R_\lambda^0][R_\eta^k - R_\eta^0][I - (\eta - \lambda))R_\eta^k]^{-1}$$

Since $\|R_\eta^k - R_\eta^0\|_2 \stackrel{k\to\infty}{\longrightarrow} 0$ we know that $\|R(\frac{1}{\eta-\lambda}, R_\eta^k) - R(\frac{1}{\eta-\lambda}, R_\eta^0)\|_2 \stackrel{k\to\infty}{\longrightarrow} 0$ uniformly for $\lambda \in \Gamma$, see [17]. This is equivalent to $\|[I - (\eta - \lambda))R_\eta^k]^{-1} - [I - (\eta - \lambda))R_\eta^0]^{-1}\|_2 \stackrel{k\to\infty}{\longrightarrow} 0$ uniformly for $\lambda \in \Gamma$ and therefore $\|[I - (\eta - \lambda))R_\eta^k]^{-1}\|_2$ is uniformly bounded for $\lambda \in \Gamma$.

Moreover, the function $\lambda \to R_\lambda^0 \in \mathcal{L}(L^2(B_1), H_0^1(B_1)) \hookrightarrow \mathcal{L}(H_0^1(B_1), H_0^1(B_1))$ is continuous, actually analitic, in $\rho(L_0)$. Therefore,

$$\|R_\lambda^k - R_\lambda^0\|_{2,1} \leq (1 + |\eta - \lambda|\|R_\lambda^0\|_1) \cdot \|R_\eta^k - R_\eta^0\|_{2,1} \cdot \|[I - (\eta - \lambda))R_\eta^k]^{-1}\|_2 \stackrel{k\to\infty}{\longrightarrow} 0$$

uniformly in $\lambda \in \Gamma$. This shows part (ii).

To show part (i) we just need to realize that if Γ is a Jordan curve enclosing a spectral set $\Sigma_0 \subset \sigma(L_0)$, then for k large Γ will enclose a spectral set $\Sigma_k \subset \sigma(L_k)$ and the projections associated to this spectral sets are given by

$$P_{\Gamma,L_0} = \frac{1}{2\pi i}\int_\Gamma R_\lambda^0 d\lambda, \qquad P_{\Gamma,L_k} = \frac{1}{2\pi i}\int_\Gamma R_\lambda^k d\lambda.$$

Hence,

$$\|P_{\Gamma,L_0} - P_{\Gamma,L_k}\|_{2,1} \leq \frac{1}{2\pi}|\Gamma| \sup_{\lambda\in\Gamma} \|R_\lambda^k - R_\lambda^0\|_{2,1} \stackrel{k\to\infty}{\longrightarrow} 0$$

This proves the proposition.

5 On the Metric.

In this Section we study how the metrics d_p^M depend on the operator M. Also, at the end of the section we give a proof of Theorem 1.1.

Let us start with the following,

Lemma 5.1 *Let $\{\Omega_k\}_{k\geq 0}$ be a family of open sets in Θ. Let $L \in \mathcal{S}$ and assume that, if $\{(\lambda_n^k, \psi_n^k)\}_{n=1}^\infty$, $k \geq 0$, are the eigenvalues and eigenfunctions, ordered and counting multiplicity, of $-L$ in Ω_k with Dirichlet boundary conditions then $\lambda_n^k \stackrel{k\to\infty}{\longrightarrow} \lambda_n^0$ and $\|\psi_n^k - \psi_n^0\|_{L^2(B_1)} \stackrel{k\to\infty}{\longrightarrow} 0$. If $\phi_k \in H_0^1(\Omega_k)$ and if $\phi_k \stackrel{k\to\infty}{\longrightarrow} \gamma$ weakly in $H_0^1(B_1)$ then $\gamma \in H_0^1(\Omega_0)$*

Proof. Without loss of generality we can assume that $\lambda_n^k > 0$ for all n and k. If this is not the case we just add a large enough constant η to the operator $-L$ and work with $\eta - L$.

Since $\phi_k \stackrel{k\to\infty}{\longrightarrow} \gamma$ weakly in $H_0^1(B_1)$ then $\|\phi_k\|_{H_0^1(B_1)} \leq C$, for certain constant independent of k, and $\phi_k \stackrel{k\to\infty}{\longrightarrow} \gamma$ strongly in $L^2(B_1)$. Notice that, since M is selfadjoint, the eigenfunctions form an orthonormal basis in $L^2(\Omega_k)$ and in $H_0^1(\Omega_k)$. Hence, $\phi_k = \sum_{n=1}^\infty (\phi_k, \psi_n^k)\psi_n^k$ and $\sum_{n=1}^\infty |(\phi_k, \psi_n^k)|^2\lambda_n^k \leq C^2$. Since $(\phi_k, \psi_n^k) \stackrel{k\to\infty}{\longrightarrow} (\gamma, \psi_n^0)$ and $\lambda_n^k \stackrel{k\to\infty}{\longrightarrow} \lambda_n^0$, we get that $\sum_{n=1}^\infty |(\gamma, \psi_n^0)|^2\lambda_n^0 \leq C^2$.

Hence $\gamma_{|\Omega_0} = \sum_{n=1}^{\infty}(\gamma, \psi_n^0)\psi_n^0 \in H_0^1(\Omega_0)$. Let us see now that $\|\phi_k\|_{L^2(\Omega_k)} \stackrel{k\to\infty}{\longrightarrow} \|\gamma_{|\Omega_0}\|_{L^2(\Omega_0)}$. If this is true, since $\|\phi_k\|_{L^2(\Omega_k)} \stackrel{k\to\infty}{\longrightarrow} \|\gamma\|_{L^2(B_1)}$, we will have that $\|\gamma\|_{L^2(B_1)} = \|\gamma_{|\Omega_0}\|_{L^2(\Omega_0)}$ and therefore $\gamma = 0$ in $B_1 \setminus \Omega_0$. This means that $\gamma \in H_0^1(\Omega_0)$.

Since L has compact resolvent we know that $\lambda_n \stackrel{n\to\infty}{\longrightarrow} +\infty$. Let ϵ be a small number and choose n_0 large enough such that $\lambda_n^0 > 1/\epsilon$ for all $n \geq n_0$. Since $\lambda_{n_0}^k \stackrel{k\to\infty}{\longrightarrow} \lambda_{n_0}^0$ we have $\lambda_{n_0}^k > 1/\epsilon$ for $k \geq k_0$. Then,

$$\begin{aligned}
|\|\phi_k\|_{L^2} - \|\gamma_{|\Omega_0}\|_{L^2(\Omega_0)}| &= |\textstyle\sum_{n=1}^{\infty} |(\phi_k, \psi_n^k)|^2 - \sum_{n=1}^{\infty} |(\gamma, \psi_n^0)|^2| \\
&\leq |\textstyle\sum_{n=1}^{n_0} |(\phi_k, \psi_n^k)|^2 - \sum_{n=1}^{n_0} \|(\gamma, \psi_n^0)|^2| + \\
&\quad + \epsilon \textstyle\sum_{n=n_0+1}^{\infty} |(\phi_k, \psi_n^k)|^2 \lambda_n^k + \epsilon \sum_{n=1}^{n_0} \|(\gamma, \psi_n^0)|^2 \lambda_n^0 \\
&\leq |\textstyle\sum_{n=1}^{n_0} |(\phi_k, \psi_n^k)|^2 - \sum_{n=1}^{n_0} \|(\gamma, \psi_n^0)|^2| + \epsilon C^2 + \epsilon C^2 \stackrel{k\to\infty}{\longrightarrow} 2\epsilon C^2
\end{aligned}$$

Since ϵ is arbitrarily small, we get $|\|\phi_k\|_{L^2} - \|\gamma_{|\Omega_0}\|_{L^2(\Omega_0)}| \stackrel{k\to\infty}{\longrightarrow} 0$. This proves the lemma.

We can prove now the following result

Proposition 5.2 *Let $M \in \mathcal{S}_0$ and $\tilde{M} \in \mathcal{D}_0$. Then*

$$u^{\Omega_k, M} \stackrel{k\to\infty}{\longrightarrow} u^{\Omega_0, M} \text{ in } L^2(B_1) \Longrightarrow u^{\Omega_k, \tilde{M}} \stackrel{k\to\infty}{\longrightarrow} u^{\Omega_0, \tilde{M}} \text{ in } L^2(B)$$

Proof. To simplify the notation let us denote by $u^k = u^{\Omega_k, M}$ and by $\tilde{u}^k = u^{\Omega_k, \tilde{M}}$, $k \geq 0$. ¿From the fact that $u^k \stackrel{k\to\infty}{\longrightarrow} u^0$ in $L^2(B_1)$ and Lemma 4.1 we get that $u^k \stackrel{k\to\infty}{\longrightarrow} u^0$ in $H_0^1(B_1)$. Also, from Lemma 2.2, we know that $\|\tilde{u}^k\|_{H_0^1(B)} \leq C$ with C independent of k. This implies that we can get a subsequence that we still denote by $\tilde{u}^k$ and a function $v \in H_0^1(B)$ such that $\tilde{u}^k \to v$ weakly in $H_0^1(B)$ and strongly in $L^2(B_1)$. From Lemma 5.1, and Proposition 4.2 we have that $v \in H_0^1(\Omega_0)$. Let us see that $v = \tilde{u}^0$. For this, let $\phi \in C_0^\infty(\Omega_0)$. Since $\text{supp}(\phi) \subset\subset \Omega_0$, we have that there exists an $\alpha > 0$ such that $\tilde{u}^0 > \alpha$ in $\text{supp}(\phi)$. Define the function $\phi_k = \phi \cdot \frac{u^k}{u^0} \in H_0^1(\Omega_0) \cap H_0^1(\Omega_k)$, for k large enough, and satisfies $\phi_k \stackrel{k\to\infty}{\longrightarrow} \phi$ in $H^1(\Omega_0)$. We have,

$$-\int_{\Omega_0} \tilde{a}_{ij}\partial_i(\tilde{u}^k)\partial_j(\phi_k) + \int_{\Omega_0} \tilde{b}_i\partial_i(\tilde{u}^k)\phi_k \stackrel{k\to\infty}{\longrightarrow} -\int_{\Omega_0} \tilde{a}_{ij}\partial_i(v)\partial_j(\phi) + \int_{\Omega_0} \tilde{b}_i\partial_i(v)\phi$$

This implies that the function $v \in H_0^1(\Omega_0)$ satisfies

$$-\int_{\Omega_0} \tilde{a}_{ij}\partial_i(v)\partial_j(\phi) + \int_{\Omega_0} \tilde{b}_i\partial_i(v)\phi = -\int_{\Omega_0} \phi$$

for any $\phi \in C_0^\infty(\Omega_0)$. This means that $v = \tilde{u}^0$. This concludes the proof of the proposition.

Corollary 5.3 *Let $M, \tilde{M} \in \mathcal{S}_0$. Then*

$$u^{\Omega_k,M} \overset{k\to\infty}{\longrightarrow} u^{\Omega_0,M} \text{ in } L^2(B_1) \Longleftrightarrow u^{\Omega_k,\tilde{M}} \overset{k\to\infty}{\longrightarrow} u^{\Omega_0,\tilde{M}} \text{ in } L^2(B)$$

Proof. Trivial from the above proposition.

Remark 5.4 *In particular, all the metrics $d^M_{L^p}$, $1 \le p < \infty$, $d^M_{H^1}$ $M \in \mathcal{S}_0$ induce the same topology in $\tilde{\Theta}$.*

We can prove now Theorem 1.1

Proof of Theorem 1.1. The fact that (i) implies (ii) is a consequence of Proposition 5.2 and Proposition 4.2.

Assume now that L is a selfadjoint operator. If u^k are the solutions of $\Delta u^k = -1$ in Ω_k with Dirichlet boundary conditions, we know that there exists a constant C such that $\|u^k\|_{H_0^1(B_1)} \le C$. Hence, we can get a subsequence, denoted again by u^k, and a function $v \in H_0^1(B_1)$ such that $u^k \overset{k\to\infty}{\longrightarrow} v$ weakly in $H_0^1(B_1)$ and strongly in $L^2(B_1)$. From Lemma 5.1, we obtain that $v \in H_0^1(\Omega_0)$. For $\phi \in C_0^\infty(\Omega)$ we denote by $P_n^k(\phi) \in H_0^1(\Omega_k)$ the projection of ϕ over the eigenspace associated to $\lambda_n^k \in \sigma(L_k)$. Notice that from (ii), we get $\|P_n^k(\phi) - P_n^0(\phi)\|_{H_0^1(B_1)} \overset{k\to\infty}{\longrightarrow} 0$. Moreover, $\int_{B_1} \nabla u^k \nabla P_n^k(\phi) = \int_{\Omega_k} \nabla u^k \nabla P_n^k(\phi) = \int_{\Omega_k} P_n^k(\phi) = \int_{B_1} P_n^k(\phi)$ and letting $k \to \infty$, we obtain, $\int_{\Omega_0} \nabla v \nabla P_n^0(\phi) = \int_{\Omega_0} P_n^0(\phi)$. Since $I = \sum_{n=1}^\infty P_n^0$, we get $\int_{\Omega_0} \nabla v \nabla \phi = \int_{\Omega_0} \phi$, which implies that $v = u^0$. This proves the theorem.

6 On the space $(\tilde{\Theta}, d_2)$

In this section we give several examples that show the convergence of domains in the metric d_2. We also show that the space $(\tilde{\Theta}, d_2)$ is not complete.

Recall that if K is a compact set in $\mathbb{R}^N$ we define the capacity of K as

$$Cap(K) = \inf\{\|\nabla\phi\|_{L^2(\mathbb{R}^N)} : \phi \in C_0^\infty(\mathbb{R}^N),\ \phi = 1 \text{ in a neighborhood of } K\}$$

It is not difficult to see that if Ω is open and $K \subset\subset \Omega$ with $Cap(K) = 0$, then $H_0^1(\Omega) = H_0^1(\Omega \setminus K)$.

Example 1. Let Ω be a domain (or an open set). Let $K \subset\subset \Omega$ with the property that $Cap(K) = 0$. Notice that this implies that $H_0^1(\Omega) = H_0^1(\Omega \setminus K)$. Let $\{\Omega_k\}_{k>1}$, $\Omega_k \subset \Omega$, with the property that for any compact set $J \subset \Omega \setminus K$ there exists a $k_0 = k_0(J)$ such that $J \subset \Omega_k$ for any $k \ge k_0$. A tipical example of a sequence satysfing this property is $\Omega_k = \Omega \setminus U_k$ where $U_k \subset \Omega$ is closed, $U_{k+1} \subset U_k$ and $\bigcap U_k = K$.

Let us see that $u^k \equiv u^{\Omega_k,\Delta} \stackrel{k\to\infty}{\longrightarrow} u^0 \equiv u^{\Omega,\Delta}$ in L^2. The proof is simple. Since $u^k \in H_0^1(\Omega_k) \subset H_0^1(\Omega \setminus K)$ and $\|u^k\|_{H_0^1}$ is uniformly bounded we can get a subsequence and a function $v \in H_0^1(\Omega \setminus K)$ such that $u^k \stackrel{k\to\infty}{\longrightarrow} v$ weakly in $H_0^1(\Omega \setminus K)$ and strongly in L^2. With the weak formulation of the equation $\Delta u = -1$ we can get that $v \in H_0^1(\Omega \setminus K)$ is the solution of $\Delta v = -1$ in $\Omega \setminus K$. Since $H_0^1(\Omega) = H_0^1(\Omega \setminus K)$ we have that $v = u^0$.

Moreover, using standard interior estimates, see for example [12], $u^k \stackrel{k\to\infty}{\longrightarrow} u^0$ in $W^{2,p}_{loc}(\Omega \setminus K)$ and therefore in $L^\infty_{loc}(\Omega \setminus K)$.

In the case where the set K is a single point, and therefore $Cap(K) = 0$, if $N \geq 2$, this case includes the one treated in [24].

Example 2. With a slight modification of the proof of Example 1 and taking into account Example 2.18 of [3], it is not difficult to prove that if Ω satisfies an exterior cone condition at each point of its boundary (not necessarily a uniform cone condition) and if $K \subset\subset \Omega$ with $Cap(K) = 0$, then if Ω_k satisfies:
(i). If $J \subset\subset \Omega \setminus K$, there exists a $k_0 = k(J)$ such that $J \subset \Omega_k$ for $k \geq k_0$
(ii). $|\Omega_k \setminus \Omega| \stackrel{k\to\infty}{\longrightarrow} 0$,
then $u^k \stackrel{k\to\infty}{\longrightarrow} u^0$ in L^2 and in $L^\infty_{loc}(\Omega \setminus K)$.

This kind of perturbation is even more general than the one given in [5]. It also includes the case of a dumbbell domain.

Example 3. Let us see now an example where $\Omega_0 \subset \Omega_k$, $|\Omega_k \setminus \Omega| \geq \delta > 0$ but still $u^k \stackrel{k\to\infty}{\longrightarrow} u^0$ in L^2. To simplify let us work in $\mathbb{R}^2$.

Let $\Omega_0 = (-1,0) \times (0,1)$ and define $Q = (-1,1) \times (0,1)$. For any $k > 1$ and any $h \in \{1, 2, \cdots k-1\}$ define the line segment $L_h^k = \{(x, \frac{h}{k});\, x \in (0,1]\}$. Let $\Omega_k = Q \setminus \bigcup_{h=1}^{k-1} L_h^k$. Then, $u^k \stackrel{k\to\infty}{\longrightarrow} u^0$ in L^2.

Again, using the fact that u^k is a bounded sequence in $H_0^1(Q)$ we can get a subsequence and a function $v \in H_0^1(Q)$ such that $u^k \stackrel{k\to\infty}{\longrightarrow} v$ weakly in $H_0^1(Q)$ and strongly in $L^2(Q)$. Moreover v satisfies $\int_{\Omega_0} \nabla v \nabla \phi = \int_{\Omega_0} \phi$ for any $\phi \in C_0^\infty(\Omega_0)$. Let us see that $v \in H_0^1(\Omega_0)$, which in turn will imply that $v = u^0$. To see this it is enough to prove that $v \equiv 0$ in $Q_+ = Q \cap \{x > 0\}$.

From the maximum principle we get that there exists a constant $C > 0$ such that $0 \leq u^k \leq C$ for all k. Define $Q_h^k = \{(x,y); 0 < x < 1,\, \frac{h}{k} < y < \frac{h+1}{k}\}$, and let w_h^k be the solution of $\Delta w = -1$ in Q_h^k with boundary conditions $w = 0$ in $\partial Q_h^k \cap \{x > 0\}$ and $w = C$ in $\partial Q_h^k \cap \{x = 0\}$. Notice that up to a translation $w_h^k = w_l^k$ for any $h, l \in \{1, 2, \cdots, n-1\}$. By the maximum principle again, we get that $u^k \leq w_h^k$ in Q_h^k. With local estimates at the boundary, see for example Theorem 9.26 of [12], we get that for any $\alpha > 0$, $\sup\{w_h^k(x,y) : x > \alpha\} \stackrel{k\to\infty}{\longrightarrow} 0$ uniformly in h. This implies that $v \equiv 0$ in Q^+ and therefore $v = u^0$.

Finally, we can show

Proposition 6.1 *If $1 \le p < \infty$, the space $(\tilde{\Theta}, d_p^{\triangle})$ is not complete.*

Proof. We will construct a cauchy sequence of open sets $\{\Omega_k\}$ in $(\tilde{\Theta}, d_p^{\triangle})$, which is not convergent.

First of all we make the observation that if Ω_k is a sequence with $\Omega_{k+1} \subset \Omega_k$ and if $\Omega_k \stackrel{k\to\infty}{\longrightarrow} \Omega_0$ then necessarily we have $\Omega_0 \subset \cap \Omega_k$ for any k. This is trivial since for fixed h, $u^k \equiv u^{\Omega_k, \triangle} \in H_0^1(\Omega_h)$ for all $k \ge h$. This means that $u^0 \in H_0^1(\Omega_h)$ and therefore $\Omega_0 \subset \Omega_h$ for any h.

We will work in $I\!R^2$. Define again $Q = (-1,1) \times (0,1)$, $Q_+ = Q \cap \{x > 0\}$ and $Q_- = Q \cap \{x < 0\}$. Let $\{q_n\}_{n=1}^{\infty}$ be an ordering of the points with rational coordinates of Q_-. Notice that the biggest open set contained in $Q \setminus \bigcup_{n=1}^{\infty} q_n$ is Q_+. Denote by $u = u^Q$ and by $u^+ = u^{Q_+}$. Since Q_+ is a proper subset of Q then $u^+ < u$ in Q_+. Let $B_0 \subset\subset Q_+$ be a small ball with the property that there exists an $\alpha > 0$ such that $u(x) - u^+(x) > 2\alpha$ for any $x \in B_0$.

In view of Example 1 and Lemma 4.1, if $Q_\epsilon = Q \setminus B(q_1, \epsilon)$ we have that $u^{Q_\epsilon} \stackrel{\epsilon\to 0}{\longrightarrow} u$ in L^p and in $L^\infty_{loc}(Q_+)$. Choose ϵ small enough so that $B(q_1,\epsilon) \cap \bar{Q}_+ = \emptyset$ and such that $\|u^{Q_\epsilon} - u\|_{L^p} \le \frac{\alpha}{2}$ and $\|u^{Q_\epsilon} - u\|_{L^\infty(B_0)} \le \frac{\alpha}{2}$. For this ϵ fixed, define $\Omega_1 = Q \setminus \bar{B}(q_1, \epsilon)$.

Let q_{2^*} be the first point in $\{q_n\}$ which is contained in Ω_1. As before, we choose an ϵ small enough such that $B(q_{2^*}, \epsilon) \cap Q_+ = \emptyset$ and with the property that if $\Omega_2 = \Omega_1 \setminus \bar{B}(q_{2^*}, \epsilon)$ then $\|u^1 - u^2\|_{L^p} \le \frac{\alpha}{2^2}$ and $\|u^1 - u^2\|_{L^\infty(B_0)} \le \frac{\alpha}{2^2}$.

We can proceed now to construct the whole sequence Ω_k, where $\Omega_k = \Omega_{k-1} \setminus \bar{B}(q_{k^*}, \epsilon)$ so that q_{k^*} is the first point in the ordering such that $q_{k^*} \in \Omega_{k-1}$, and ϵ is small enough so that $B(q_{k^*}, \epsilon) \cap Q_+ = \emptyset$ and $\|u^k - u^{k-1}\|_{L^p} \le \frac{\alpha}{2^k}$, $\|u^k - u^{k-1}\|_{L^\infty(B_0)} \le \frac{\alpha}{2^k}$.

It is clear that the sequence $\{u^k\}$ is a Cauchy sequence in $L^p(Q)$. It will converge to a function $v \in L^p(Q)$. Moreover we get that $\|v - u\|_{L^\infty(B_0)} \le \sum_{k=1}^{\infty} \frac{\alpha}{2^k} = \alpha$. This implies that $v(x) - u^+(x) > \alpha$ for any $x \in B_0$. If there exists an open Ω_0 such that $d_p^{\triangle}(\Omega_k, \Omega_0) \stackrel{k\to\infty}{\longrightarrow} 0$ then, from the observation above, necessarily we have $\Omega_0 \subset \bigcap \Omega_k \subset Q \setminus \bigcup q_n$, which implies that $\Omega_0 \subset Q_+$. In particular $u^{\Omega_0} \le u^+$ and therefore $v \ne u^{\Omega_0}$. This shows that the sequence $\{\Omega_k\}$ is not convergent in $d_p^{\triangle}$. This concludes the proof of the proposition.

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