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Semilinear Elliptic Systems: A survey of superlinear problems [1]

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Abstract: In this paper we survey recent results on existence of solutions for elliptic Hamiltonean systems. Both bounded domains and R^N are considered. We discuss existence of solutions, decay and symmetry.

Key words: Elliptic systems, Variational Methods, A priori estimates.

§1. Introduction.

In this paper we consider a system of partial differential equations of elliptic type

$$(S) \qquad \begin{cases} -\Delta u = f(x,u,v) \\ -\Delta v = g(x,u,v) \end{cases}$$

This kind of systems appears in many different branches of Physics, Biology, ... There is a large literature devoted to a variety of problems posed to system (S), such as boundary value problems, maximum principles, eigenvalue problems, local behavior of solutions, global properties of solutions (decay, symmetry), existence of ground states, etc. We propose to discuss here only some of these problems, as they are related to our research in recent years. At the outset, we warn the reader that the following questions will not be considered here: sublinear problems, maximum principles for cooperative and noncooperative systems, critical problems, problems in R^2 and local behavior of solutions.

Here we study two main problems:

i) Existence of (positive) solutions for the Dirichlet problem in bounded domains Ω of R^N, $N \geq 3$.

ii) Existence of positive solutions defined in the whole of R^N, $N \geq 3$.

These two problems will be studied for superlinear problems, and in this process we come to other matters, such as decay of solutions at ∞ and symmetry properties.

There are two main classes of systems (S) that can be treated variationally: gradient systems and Hamiltonean systems.

System (S) is said to be *gradient* if there exists a function $F(x,u,v)$ such that $f = \dfrac{\partial F}{\partial u}$ and $g = \frac{\partial F}{\partial v}$. In this case, the functional associated to the system has the form

$$\Phi(u,v) = \frac{1}{2}\int(|\nabla u|^2 + |\nabla v|^2) - \int F(x,u,v)$$

and the appropriate space to consider this functional over is the Cartesian product $H^1 \times H^1$ or some other product of H^1-spaces depending on the boundary conditions.

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This case has been studied by several authors; to mention some Bartsch [B], Costa-Magalhães [CM1] [CM2] [CM3].

When we say that the functional Φ is appropriate to the systems (S), we simply mean that the Euler Lagrange equations associated to Φ are the ones in systems (S). That is, the critical points of Φ are weak solutions of (S) in a sense which is made precise at each problem.

System (S) is said to be *Hamiltonean* if there exists a function $H(x, u, v)$ such that $f = \dfrac{\partial H}{\partial v}$ and $g = \dfrac{\partial H}{\partial u}$. In this case, it would appear natural to associate to the system the functional

$$\Phi(u, v) = \int \nabla u \nabla v - \int H(x, u, v), \tag{1}$$

whose quadratic part (the part involving the gradient) is well defined if we work again with a product of H^1 spaces. However, we now explain why this is not the right functional in general. This is due to the fact that we want to consider problems where the growth of f and g with respect to u and v have different speeds at ∞. We next explain this.

A model example for a Hamiltonean system is the following

$$\begin{cases} -\Delta u = |v|^{p-1} v \\ -\Delta v = |u|^{q-1} u \end{cases} \tag{2}$$

whose Hamiltonean function is

$$H(u, v) = \frac{1}{p+1}|v|^{p+1} + \frac{1}{q+1}|u|^{q+1}. \tag{3}$$

So, if we work with $H^1 \times H^1$, the second term in the functional Φ, given in (1), is well defined if $p, q \leq (N+2)/(N-2)$. This is due to the Sobolev imbedding $H^1 \subset L^r, 2 \leq r \leq 2N/(N-2)$ for $N \geq 3$. However, in the example

$$\begin{cases} -\Delta u = v \\ -\Delta v = |u|^{q-1} u, \end{cases} \tag{4}$$

which is equivalent (speaking just of the equations) to the biharmonic

$$\Delta^2 u = |u|^{q-1} u,$$

the proper limitation for q is $q \leq (N+4)/(N-4)$.

We observe that the Dirichlet boundary value problem for the above system, $u = v = 0$ or $\partial\Omega$, corresponds to Navier boundary value problem for the biharmonic, $u = \Delta u = 0$ on $\partial\Omega$.

These remarks lead us to the conclusion that u and v have to be taken in distinct Sobolev spaces. Precisely, if $p < (N+2)/(N-2)$, one may take u in some space smaller that H^1, and v in some space larger that H^1, allowing us to take $q > (N+2)/(N-2)$. For this reason we shall consider fractional Sobolev spaces in the sequel.

Also, this interplay between p and q is put in correct terms by the introduction of the so-called critical hyperbola, a concept introduced independently by Clément-de Figueiredo - Mitidieri [CFM1] and by Peletier-van der Vorst [PV].

Definition of the critical hyperbola. The pair $(p,q) \in R^2$ is in the critical hyperbola if $p, q > 0$ and

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}. \tag{5}$$

Observe that, if $p = q$ and (p,q) is in the critical hyperbola, we have

$$p = q = (N+2)/(N-2),$$

which is precisely the critical exponent of the scalar problem (i.e., a single equation).

So, for the system (S), the critical hyperbola replaces the notion of critical exponent which appears in the scalar case.

Definition of subcritical problem. Let p be the v-growth of $f(x,u,v)$ and q be the u-growth of $g(x,u,v)$. A system (S) with such f and g is called *subcritical* if

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}. \tag{6}$$

As we shall see later, the solvability of such a system is going to depend on the u-growth of f and on the v-growth of g.

Definition of superlinear system. Let p and q be as in the previous definition. The corresponding system (S) is *superlinear* if

$$1 > \frac{1}{p+1} + \frac{1}{q+1} \tag{7}$$

A condition like (7) implies that both p and q cannot be both ≤ 1. As a matter of fact, we assume that both are ≥ 1, and at least one of them > 1.

§2. Existence of positive solutions for subcritical problems via degree theory.

In this section we consider the Dirichlet problem for the system

$$\begin{cases} -\Delta u = f(v) \\ -\Delta v = g(u) \end{cases} \tag{8}$$

in a bounded domain Ω. Let us describe the ideas for the solvability of (8) in the case that $(0,0)$ is a trivial solution and we look for a nontrivial solution.

Suppose then that $f(0) = g(0) = 0$ and $f(t) \geq 0$ and $g(t) \geq 0$ for $t > 0$. Let S be the inverse of the Laplacian subject to Dirichlet boundary condition, that is $S : C(\overline{\Omega}) \to C(\overline{\Omega})$, where $C(\overline{\Omega})$ is the space of continuous functions in $\overline{\Omega}$, defined as follows: for $u \in C(\overline{\Omega})$, Su is the solution ω of

$$-\Delta\omega = u \quad \text{in} \quad \Omega, \ \omega = 0 \quad \text{on} \quad \partial\Omega.$$

Such a problem has a unique solution $\omega \in C^1(\overline{\Omega})$. So, system (8) is equivalent to

$$u = Sf(Sg(u)), \qquad v = Sg(u) \tag{9}$$

Hence, the problem is to find a fixed point of the (nonlinear) operator

$$T(\cdot) = Sf(Sg(\cdot)) \tag{10}$$

which can be considered acting in the cone of nonnegative continuous functions. This is due to the fact that S is a positive linear operator, as a consequence of the maximum principle. In order to obtain a nontrivial fixed point of T, we use the following result, which is basically due to M.A. Krasnoselskii [K] and is given here in a form obtained by Benjamin [B].

Proposition 2.1. Let C be a cone in a Banach space X and $\Phi : C \to C$ a compact map such that $\Phi(0) = 0$. Assume that there exist numbers $0 < r < R$ and a vector $v \in C \setminus \{0\}$ such that (i) $x \neq t\Phi(x)$ for $0 \leq t \leq 1$ and $||x|| = r$, and (ii) $x \neq \Phi(x) + tv$ for $t \geq 0$ and $||x|| = R$. Then, if $U = \{x \in C : r < ||x|| < R\}$ and $B_\rho = \{x \in C : ||x|| < \rho\}$, one has

$$i_C(\Phi, B_R) = 0, \quad i_C(\Phi, B_r) = 1 \quad \text{and} \quad i_C(\Phi, U) = -1.$$

In particular, Φ has a fixed point in U. Condition (i) above is satisfied if there is a bounded linear operator $A : X \to X$, such that $A(C) \subset C$, A has spectral radius < 1 and $\Phi(x) \leq Ax$ for $x \in C$, $||x|| = r$. Here i_C denotes the Leray Schauder index.

In order to establish (ii) for our operator T we need to prove a priori bounds for the positive solutions of system (8). So let us now be more precise about the hypotheses that f and g should satisfy.

(H1) $f, g : R^+ \to R$ are $C^1, f(0) = g(0) = 0, f'(t), g'(t) \geq 0$.

(H2) There are numbers $a, b \in]0, \infty]$, with $ab > \lambda_1^2$, where λ_1 is the first eigenvalue of $(-\Delta, H_0^1)$, such that

$$\liminf_{s \to +\infty} \frac{f(s)}{s} \geq a, \qquad \liminf_{s \to +\infty} \frac{g(s)}{s} \geq b$$

(H3) There are constants c and numbers p, q satisfying (6) and (7) such that

$$f(t) \leq c(t^p + 1), \qquad g(t) \leq c(t^q + 1), \quad \forall t \geq 0$$

A priori bounds for positive solutions of system (8) have been proved in [CFM1] for convex domains, and in the case of the ball and radial symmetric solutions in [PV]. The next result is from [CFM1], whose proof uses moving parallel planes, or more precisely, Troy [T] generalization of the result of Gidas-Ni-Nirenberg [GNN]. We remark that these results using moving planes have had their proofs simplified in Berestycki-Nirenberg [BN], using Varadhan's maximum principle. Also Troy's result has been proved in [dF]using this maximum principle.

Theorem 2.2. Assume (H1), (H2), (H3) above, and that Ω is convex with $\partial\Omega \in C^3$. Then there is a constant $C > 0$ such that

$$||u||_{L^\infty}, ||v||_{L^\infty} \leq C$$

for all positive solutions (u, v) of system (8).

Once a priori bounds are established, we can use Proposition 2.1 and prove the existence of nontrivial solutions of system (8). We have the following result from [CFM1].

Theorem 2.3. Assume (H1), (H2), (H3) and that $f'(0)g'(0) < \lambda_1^2$. Suppose Ω convex, with $\partial\Omega \in C^3$. Then system (8) has at least one positive solution $(u, v), u, v \in C^{2,\alpha}(\overline{\Omega}),\ 0 < \alpha < 1$.

§3. Existence of solutions for Hamiltonean systems via variational methods.

In this section we shall consider general Hamiltonian systems, which have the form

$$\begin{cases} -\Delta u = H_v(x, u, v) \\ -\Delta v = H_u(x, u, v) \end{cases} \tag{11}$$

subject to Dirichlet boundary conditions. The basic assumptions on the Hamiltonean are

(H1) $H : \overline{\Omega} \times R^2 \to R^+ \cup \{0\}$ is of class C^1.

(H2) There exists a constant $c > 0$ such that

$$|H_u(x, u, v)| \leq c(|u|^p + |v|^{\frac{p(q+1)}{p+1}} + 1)$$
$$|H_v(x, u, v)| \leq c(|v|^q + |u|^{\frac{q(p+1)}{q+1}} + 1)$$

where (p, q) satisfy relations (6), (7). That is, our system is superlinear and subcritical.

(H3) There exists $R > 0$ such that
$\frac{1}{p+1} H_u u + \frac{1}{q+1} H_v v \geq H$ for all $x \in \overline{\Omega}$, and $|(u, v)| \geq R$.

(H4) There are constants $r > 0$ and $c > 0$ such that
$H(x, u, v) \leq c(|u|^\alpha + |v|^\beta)$ for all $x \in \overline{\Omega}$, and $|(u, v)| \leq r$.
where $\alpha, \beta \geq 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} < 1$.

Existence of solutions for Hamiltonean systems by variational methods has been studied by de Figueiredo-Felmer [FF1] and by Hulshof-vander Vorst [HV], and more recently by Costa-Magalhães [CM] and de Figueiredo-Magalhães [FMa]. The critical case has been studied quite recently by Costa-Silva [CS] and Hulshof-Mitidieri-vander Vorst [HMV], using difference methods.

As said before, since one of the p or q could be larger than $(N+2)/(N-2)$, we cannot work in $H_0^1 \times H_0^1$. We need fractional Sobolev spaces, which we now introduce.

Let $0 < \lambda_1 < \lambda_2 \leq \ldots$ be the e-values of $(-\Delta, H_0^1(\Omega))$ and $\varphi_1, \varphi_2, \ldots$ corresponding e-functions. For $0 \leq s \leq 2$, define the following subspace of $L^2(\Omega)$

$$E^s = \{u \in \sum_{n=1}^{\infty} a_n \varphi_n \in L^2 : \sum \lambda_n^s a_n^2 < \infty\}.$$

which is a Banach space with norm defined by

$$||u||_{E^s}^2 = \sum_{n=1}^{\infty} \lambda_n^s a_n^2.$$

It is easy to see that

$$A^s : E^s \to L^2$$

is an isomorphism. Also it can be proved the Sobolev imbedding theorem:

"E^s is continuously imbedded in L^σ for $\frac{1}{\sigma} \geq \frac{1}{2} - \frac{s}{N}$, and compactly if there is strict inequality". For references to these questions see [FF1].

Now, using these fractional Sobolev spaces, we introduce the appropriate functional to be associated to system (9), when we assume (H2). Choose $s + t = 2, s, t > 0$, such that

$$\frac{1}{p+1} > \frac{1}{2} - \frac{s}{N}, \quad \frac{1}{q+1} > \frac{1}{2} - \frac{t}{N}.$$

Then, define $\Phi : E^s \times E^t \to \mathbb{R}$ by

$$\Phi(u,v) = \int A^s u A^t v - \int H(x,u,v), \tag{12}$$

which is well defined and of class C^1. Also, the critical points of Φ are weak solutions of system (9) in the following sense: $(u,v) \in E^s \times E^t$ and

$$\int A^s u A^t \psi - \int H_v(x,u,v)\psi = 0, \qquad \forall \psi \in E^t$$
$$\int A^s \phi A^t v - \int H_u(x,u,v)\phi = 0, \qquad \forall \phi \in E^s.$$

It can be shown that every weak solutions in the above sense is a strong solution, i.e. $u \in W^{2,\frac{p+1}{p}} \cap W_0^{1,\frac{p+1}{p}}$, $v \in W^{2,\frac{q+1}{q}} \cap W_0^{1,\frac{q+1}{q}}$. See [FF1].

In the search of critical points of (10) we are reassured by the fact that the nonlinear part has some compactness in view of the fact that E^s is compactly imbedded in L^{p+1}, and E^t in L^{q+1}. However, the functional is strongly indefinite, which means that it is coercive in a subspace of $E = E^s \times E^t$ with infinite dimension and it is anti-coercive in another subspace of E, which is also of infinite dimension. We have to recourse to a liking theorem of Benci-Rabinowitz [BR] in a version due to Felmer [F]. Namely,

Theorem 3.1. Let $\Phi : E \to I\!R$ a functional having the form

$$\Phi(z) = \frac{1}{2}\langle Lz, z\rangle + \mathcal{H}(z)$$

where E is a Hilbert space (inner-product $\langle,\rangle$), which is a direct sum of two spaces X and Y not necessarily orthogonal. Assume that
(I1) $L : E \to E$ is a linear bounded self-adjoint operator,
(I2) $\mathcal{H}'$ is compact,
(I3) There are linear bounded invertible operators $B_1, B_2 : E \to E$ such that if $\omega \in R_0^+$, then the linear operator

$$\widehat{B}(\omega) := P_X B_1^{-1} \exp(\omega L) B_2 : X \to X$$

is invertible, where P_X is the projection of E over X along Y.

Let

$$S = \{B_1 z : ||z|| = \rho, z \in Y\}$$

and for $z^+ \in Y, z^+ \neq 0,\ \sigma > \rho/||B_1^{-1}B_2 z^+||$ and $M > \rho$, define

$$Q = \{B_2(\tau z^+ + z) : 0 \le \tau \le \sigma, ||z|| \le M,\ z \in X\}.$$

The sets S and ∂Q (the relative boundary of Q) link. Assume that there a constant $\delta > 0$ such that

(IS) $\Phi(z) \geq \delta,\ z \in S$
(IQ) $\Phi(z) \leq 0,\ z \in \partial Q$.
Then Φ possesses a critical point critical value $\geq \delta$, provided Φ satisfies the (PS) condition.

In order to study the geometry of the functional Φ defined in (10) we observe that the quadratic part

$$Q(z) = \int A^s u A^t v, \qquad \text{for } z = (u, v) \in E = E^s \times E^t$$

is the quadratic form coming from the bilinear form

$$B : E \times E \to R$$

$$B[(u, v), (\phi, \psi)] = \int A^s u A^t \psi + A^s \phi A^t v,$$

and the self-adjoint bounded linear operator $L : E \to E$ associated to it, namely

$$B[z, \eta] = (Lz, \eta)_E$$

has two eigenvalues $+1$ and -1. So E is split into two subspaces E^+ and E^-, which are the correponding eigenspaces to $+1$ and -1:

$$E^+ = \{(u, A^{-t}A^s u) : u \in E^s\}$$
$$E^- = \{(u, -A^{-t}A^s u) : u \in E^s\}.$$

Now, in order to apply Theorem 3.1 we define the sets S and Q as follows. Choose $\mu, \nu > 1$ such that

$$\frac{1}{p+1} < \frac{\mu}{\mu+\nu}, \frac{1}{q+1} < \frac{\nu}{\mu+\nu}.$$

Then, for $\rho > 0$ define

$$S := \{(\rho^{\mu-1}u, \rho^{\nu-1}v) : ||(u, v)|| = \rho,\ (u, v) \in E^+\}$$

and choose $z^+ = (u^+, v^+) \in E^+$, with u^+ an eigenfunction of $(-\Delta, H_0^1)$. Then, for some $\tau, M > 0$ define

$$Q = \{\tau(\sigma^{\mu-1}u^+, \sigma^{\nu-1}v^+) + (\sigma^{\mu-1}u, \sigma^{\nu-1}v) : \\ 0 \leq \tau \leq \sigma,\ 0 \leq ||(u, v)||_E \leq M, (u, v) \in E^-\}.$$

It can be verified that under hypotheses (H1)-(H4),we have:
(i) there exists $\rho > 0$ such that $\Phi(z) \geq \delta$, for $z \in S$,
(ii) there exist $\sigma, M > 0, \sigma > \rho$, such that $\Phi(z) \leq 0$ for $z \in \partial Q$.

Finally, hypotheses (H1)-(H3) suffice to prove that Φ satisfy the (P.S.) condition. So we have the following result.

Theorem 3.2. ([FF1]) Assume (H1)-(H4). Then system (9) has at least one non-trivial solution $u \in W^{2,\frac{p+1}{p}}(\Omega) \cap W_0^{1,\frac{p+1}{p}}(\Omega), v \in W^{2,\frac{q+1}{q}}(\Omega) \cap W_0^{1,\frac{q+1}{q}}(\Omega)$.

A similar result is proved in [BR] for the case $p, q < (N+2)/(N-2)$. An extension of Theorem 3.2 to cases where there are "quadratic" terms in the right side of (9) has been done in [FMa].

§4. On a priori bounds for superlinear problems.

We learned from the scalar case (problems involving only one equation) that there are at least three methods to be used in order to get a priori bounds of positive solutions.

First, using inequalities interpolated between Hardy's inequality

$$\int_\Omega \left|\frac{u}{\varphi_1}\right|^2 \leq c \int_\Omega |\nabla u|^2, \quad u \in H_0^1(\Omega)$$

and Sobolev's inequality, $||u||_{L^\sigma} \leq c||\nabla u||_{L^2}, 2 \leq \sigma \leq 2N(N-2)$, Brézis and Turner [BT] obtained a priori estimates for positive solution of

$$-\Delta u = f(x, u) \quad \text{in} \quad \Omega, \ u = 0 \quad \text{on} \quad \partial\Omega$$

for f such that

$$|f(x,u)| \leq c(|u|^p + 1) \qquad p < (N+1)/(N-1),$$

besides some other conditions. This method has been utilized in the case of systems by Clément-de Figueiredo-Mitidieri [CFM2] and by Cordeiro [C]. In the case of systems, the exponent $(N+1)/(N-1)$ is replaced by two hyperbolas. Namely, the estimates can be obtained by this method, provided p and q satisfy

$$\frac{1}{p+1} + \frac{N-1}{N+1}\frac{1}{q+1} > 1 - \frac{2}{N+1}$$

and

$$\frac{N-1}{N+1}\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N+1}.$$

The second method is based in the moving parallel planes technique and Pohožaev-type identities. In the scalar case this was used by de Figueiredo-Lions-Nussbaum [FLN]. As said before an application of this method to some special systems was made in [CFM1].

The third one is the so-called blow-up method, which was first used by Gidas-Spruck [GS] for the scalar case. Souto [S] used this method to some classes of semilinear systems. We believe that larger classes of systems can eventually be treated by this technique.

The proof by blow-up is done by a contradiction argument. So the hypotheses that there is a sequence of solutions (let us look the scalar case as in [GS]) with L^∞-norms going to infinite leads to assertions like this:

(!) There is a positive C^2 function $u(x)$, defined in the whole of R^N, such that

$$-\Delta u = u^p,$$

where p is the growth of $f(u)$ as $u \to \infty$(!)

This is a Liouville-type problem, which we will comment in the next section.

§5. Liouville-type Theorems.

Let us start with a single equation, and let us consider the equation

$$-\Delta u = |u|^{p-1}u \quad \text{in} \quad R^N. \tag{13}$$

It is known that if $p = (N+2)/(N-2)$, $N \geq 3$, i.e. the critical exponent, then for each $\mu > 0$ and for each $a \in R^N$ the function

$$u(x) = \left[\frac{\mu\sqrt{N(N-2)}}{\mu^2 + |x-a|^2}\right]^{\frac{N-2}{2}} \tag{14}$$

is a solution of (11), and, in fact, those are all the positive solutions of (11).

The following Liouville-tye theorem was proved in [GS]: if $1 < p < (N+2)/(N-2)$, then (11) has no C^2 positive solution defined in the whole of R^N. This is a very difficult result to prove. The difficulty arises from the fact that one does not have any assumption on the behavior of u at ∞. So, a direct use of the technique of parallel planes does not work. However, it has been observed by Caffarelli-Gidas-Spruck [CGS] and Chen-Li [CL] that if we Kelvin transform

$$\omega(x) = \frac{1}{|x|^{N-2}} u\left(\frac{x}{|x|^2}\right)$$

our equation becomes

$$-\Delta\omega = \frac{1}{|x|^{N+2-p(N-2)}}|\omega(x)|^{p-1}\omega(x).$$

Now we know that ω decays at ∞ like $|x|^{2-N}$. And since the coefficient of ω is monotonically decreasing [recall that $p < (N+2)/(N-2)$], we can apply the moving plane technique.

Now let us pass to the system

$$\begin{cases} -\Delta u = |v|^{p-1}v \\ -\Delta v = |u|^{q-1}u \end{cases} \tag{15}$$

and let us ask about the existence of positive solutions in the whole of R^N. System (15) can be written (since we are dealing with positive solutions) as

$$-\Delta((-\Delta u))^{1/p}) = u^q \tag{16}$$

It has been proved by P.-L. Lions [L] that if p, q are in the critical hyperbola then

$$K_{pq} = \inf\{||\Delta u||_{L^{\frac{q+1}{q}}} : u \in D^{2,\frac{q+1}{q}}, ||u||_{L^{p+1}} = 1\}$$

is attained at a function $u \in D^{2,\frac{q+1}{q}}$, which gives rise to a solution of (14). (Here $D^{2,\sigma}$ is the completion of $C_0^\infty(R^N)$ with respect to the norm $||\Delta.||_{L^\sigma}$). Hence, in the case that (p, q) are on the critical hyperbola, the system (13) has a positive solution $(u, v), u > 0, v > 0$ on the whole of R^N.

It is an open problem, at the present, the following question: if, for all, $p, q \geq 1$ and below the critical hyperbola, system (13) does not have positive solutions. We conjecture that the Liouville-type result holds for all $p, q \geq 1$ which are below the critical hyperbola. Several partial results have been proved:

1) In [FF2] it was proved the following Liouville-type result: if $p, q \leq (N+2)/(N-2)$ and at least one of them is strictly less than $(N+2)/(N-2)$ then there are no positive C^2 solutions of (13). The idea of the proof was to use Kelvin's transform in order to obtain some behavior of the transformed solutions at ∞ and then apply the method of parallel planes. However, by the use of Kelvin transform we obtain the two equations (assume the solutions are ≥ 0) below

$$-\Delta\omega = \frac{1}{|x|^{(N+2)-p(N-2)}} z^p$$
$$-\Delta z = \frac{1}{|x|^{(N+2)-q(N-2)}} \omega^q,$$

which have coefficients with different monotonicities, if one of the powers p or q is greater than $(N+2)/(N-2)$ and the other is smaller. That is precisely why we have to restrict to both p and $q \leq (N+2)/(N-2)$, in order to use moving planes.

2) In $[JQ]$ a similar result is obtained by a different method. He uses some identities already used in [GS2].

3) Souto in [S] reduces the problem to a Liouville theorem for an inequality and proves the result for p and q such that

$$\frac{1}{p+1}+\frac{1}{q+1}\geq\frac{N-2}{N-1}.$$

We remark that although this region is below a hyperbola, which is below the critical, it contains points where p or q can be greater than $(N+2)/(N-2)$.

4) Mitidieri [M1] proved that the conjecture is true for radially symmetric functions.

5) Serrin and Zou [SZ] proved the conjecture in a class of functions which have a certain decay at ∞.

§6. System in R^N.

From now on we consider systems of the form

$$\begin{cases} -\Delta u+u=f(x,v) \\ -\Delta v+v=g(x,u) \end{cases} \tag{17}$$

in the whole of R^N. We shall discuss the following questions relative to (15): (i) decay of positive solutions as $|x|\to\infty$; (ii) symmetry properties of positive solutions; (iii) existence of radial symmetric positive solutions; (iv) existence of ground state. Let us start with the the first question. We assume the following properties on the nonlinear terms

(H1) $f,g:R^N\to R$ are continuous functions, and

$$\begin{aligned} &f(x,t)=g(x,t)=0 \qquad \text{for} \quad t\leq 0 \qquad \text{and all} \quad x\in R^N. \\ &f(x,t),g(x,t)\geq 0 \qquad \text{for} \quad t>0. \end{aligned}$$

(H2) There is a constant $c_1>0$ such that

$$f(x,t)\leq c_1(t^p+1),\;\; g(x,t)\leq c_1(t^q+1),\;\; t\geq 0,$$

where $p,q>1$ and

$$\frac{1}{p+1}+\frac{1}{q+1}>1-\frac{2}{N},\;\; N\geq 3,$$

and

$$p,q\leq (N+4)/(N-4) \qquad \text{for} \quad N\geq 5.$$

(H3) There are constants $\alpha,\beta>1$ such that

$$0<\alpha F(x,t)\leq tf(x,t),\;\; 0<\beta G(x,t)\leq tg(x,t),\;\; t>0,$$

where $\frac{1}{\alpha}+\frac{1}{\beta}<1$.
(H4) There are real numbers $a, b \geq 1$ and positive constants c_2 and r such that

$$f(x,t) \leq c_2 t^a, \quad g(x,t) \leq c_2 t^b \qquad 0 \leq t \leq r$$

Remarks. 1) Condition (H3) implies that both f, g are superlinear. Indeed it follows from (H3) that

$$F(x,t) \geq ct^\alpha, \ G(x,t) \geq ct^\beta, \ f(x,t) \geq ct^{\alpha-1}, g(x,t) \geq ct^{\beta-1},$$

for $t \geq \varepsilon$, some $\varepsilon > 0$. These inequalities can be proved to be true for all $t \geq 0$, by using (H4). A typical example of functions satisfying (H1)-(H4) is $f = (t^+)^p$ and $g = (t^+)^q$; in this case $\alpha = p+1, \beta = q+1, \ a = p, b = q$.

Theorem 6.1. Assume (H1), (H4) with $a = b = 1$, and (H2) with p, q satisfying

$$p, q < (N+2)/(N-2). \tag{18}$$

Then, the strong positive solutions of (15), $u \in W^{2,\frac{p+1}{p}}, v \in W^{2,\frac{q+1}{q}}$ are such that

$$u(x), v(x), \nabla u(x), \nabla v(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Theorem 6.2. Assume (H1), (H2) with p, q satisfying (16), and (H4) with $a, b > 1$. Then the strong positive solutions decay exponentially. More precisely, for $0 < \theta < 1$

$$u(x)e^{\theta|x|}, v(x)e^{\theta|x|} \to 0 \qquad \text{as} \quad |x| \to \infty.$$

The proofs of the above theorems rely on the following basic lemma.

Lemma 6.3. Let u, v be as in Theorem 6.1. Then, they belong to L^γ for all $\gamma \in [2, \infty)$.

Sketch of proof. From the Sobolev imbedding theorem we have

$$u \in L^\gamma(R^N) \quad \text{for} \quad \frac{p+1}{p} \leq \gamma \leq \frac{N(p+1)}{Np-2(p+1)}$$
$$v \in L^\gamma(R^N) \quad \text{for} \quad \frac{q+1}{q} \leq \gamma \leq \frac{N(q+1)}{Nq-2(q+1)}$$

We observe that then $u, v \in L^{2^*}(R^N)$, where $2^* = 2N/(N-2)$. We also observe that through a bootstrap argument (local reasoning) it follows that u and v are continuous functions.

Since we are working in R^N, we have to be careful with convergence of the integrals. So in order to proceed we do appropriate truncations. Let, for each $k > 0$,

$$\Omega_k = \{x \in R^N : |u(x)| + |v(x)| < k\}.$$

It is easy to see that given $x_0 \in R^N$, one has that

$$R(k) = \sup\{r > 0 : B_r(x_0) \subset \Omega_k\} \to \infty \qquad \text{as} \quad k \to \infty.$$

Here $B_r(x_0)$ is the open ball of radius r centered at x_0. Let $\varphi \in C_0^\infty(R^N)$ be such that

$$\begin{aligned} &\varphi(x) = 1, \quad \text{for} \quad x \in B_{1/2}(0), \quad \varphi(x) = 0 \quad \text{for} \quad x \in R^N \setminus B_1(0), \\ &\theta \le \varphi(x) \le 1 \qquad \text{and} \qquad |\nabla\varphi(x)| \le \text{ const.} \end{aligned}$$

Define $\varphi_R(x) = \varphi\left(\dfrac{x - x_0}{R}\right)$ for $R := R(k)$. Multiply the first equation in (15) by $\varphi_R^2|u|^{s-1}u$, with $s > 1$, and the second by $\varphi_R^2|v|^{s-1}v$. Then integrate by parts, use Sobolev, Hölder, Young, etc. (see details in [FY]). Letting

$$A = \left(\int (\varphi_R^2|u|^{s+1})^{\frac{N}{N-2}}\right)^{\frac{N-2}{N}}, \quad B = \left(\int (\varphi_R^2|v|^{s+1})^{\frac{N}{N-2}}\right)^{\frac{N-2}{N}}$$

we come to

$$A \le \varepsilon c A^{\frac{s}{s+1}} B^{\frac{1}{s+1}} + C(\varepsilon)\left(\int \varphi_R^2(|u|^{s+1} + |v|^{s+1})\right) + C(\varepsilon)\int |\nabla\varphi_k|^2|u|^{s+1}$$

and a similar expression for B.

Assuming that $\displaystyle\int |u|^{s+1}$ and $\int |v|^{s+1}$ are bounded we obtain

$$A \le \varepsilon c A^{\frac{s}{s+1}} B^{\frac{1}{s+1}} + c(\varepsilon), \ \ B \le \varepsilon c A^{\frac{1}{s+1}} B^{\frac{s}{s+1}} + c(\varepsilon)$$

and then

$$AB \le \varepsilon c(\varepsilon) A^{\frac{s}{s+1}} B^{\frac{1}{s+1}} + \varepsilon c(\varepsilon) A^{\frac{1}{s+1}} B^{\frac{s}{s+1}} + c(\varepsilon)$$

which implies

$$\int |u|^{(s+1)\frac{N}{N-2}}, \int |v|^{(s+1)\frac{N}{N-2}} < \infty.$$

Repeating this procedure we get that $u, v \in L^\gamma(R^N)$ for all $\gamma = (s+1)\left(\dfrac{N}{N-2}\right)^2$ provided we assume $u, v \in L^{s+1}(R^N)$. So take $s + 1 = 2^*$, since as we observed in beginning of the proof $u, v \in L^{2*}(R^N)$. The other values of γ are obtained by the Riesz-Thorin interpolation. This completes the proof of Lemma 6.3.

Proof of Theorem 6.1. From (H2) and (H4) we have

$$|g(x,v)| \leq c(|v|^p + |v|).$$

So

$$||g(x,v)||_{L^\gamma(B_2)} \leq c(||v||^p_{L^{\gamma p}(B_2)} + ||v||_{L^\gamma(B_2)}, \qquad \text{for all} \quad \gamma. \tag{19}$$

where $B_2 = B_{2R}(x_0)$. By the Calderón-Zygmund inequlity we conclude that $u \in W^{2,\gamma}(B_2)$ from the fact that $\Delta u \in L^\gamma(B_2)$ (use (17)). By interior L^p-estimates we have

$$||u||_{W^{2,\gamma}(B_1)} \leq c(||u||_{L^\gamma(B_2)} + ||g(x,v)||_{L^\gamma(B_2)}$$

where $B_1 = B_R(x_0)$. Now using (17) and Sobolev imbedding ($\gamma > N$) we get

$$||u||_{C^{1,\lambda}(\overline{B}_1)} \leq c(||u||_{L^\gamma(B_2)} + ||v||^p_{L^\gamma(B_2)} + ||v||_{L^\gamma(B_2)})$$

Letting $|x_0| \to \infty$, the result follows. So the proof of Theorem 6.1 is complete.
For the proof of Theorem 6.2 we refer to [FY].

§7. Symmetry of positive solutions.

In order to prove summetry we require stronger assumptions than (H1)-(H4). Namely.
(H5) like (H1) plus $f, g \in C'$ and f', g' monotone.
(H6) $f'(t) \leq c_1(t^{p-1}+1) \qquad g'(t) \leq c_2(t^{q-1}+1)$, wite

$$p, q < (N+2)/(N-2)$$

§8. Final remarks.

(i) Systems like

$$-\Delta u + a(x)u = f(x,v)$$
$$-\Delta v + b(x)v = g(x,v)$$

in the whole of R^N with $a, b : R^N \to R$ such that

$$a(x), b(x) > 0 \qquad \text{and} \qquad a(x), b(x) \to +\infty, \quad \text{as} \ |x| \to \infty,$$

carries a lot of compactness. This is due to the fact that operators like $-\Delta + a$, with a satisfying the above hypothesis, behave just like $-\Delta$ in bdd domains, i.e. they have a point spectrum, etc. In our case, which is $a(x) = b(x) = 1$ we gain some compactness by restricting ourselves to radially symmetric functions. By a result of Strauss [St] and later generalizations by Lions [PLL] we know that the space $W_r^{2,s}$of radial symmetric functions of $W^{2,s}(R^N)$, $s \geq 1$ are compactly imbedded in L^σ for $2 < \sigma < 2N/(N-2s)$. Then the existence of radial symmetric solutions of (15) follows a reasoning analogous to the approach in [FF1]; see [FY].

(ii) Ground state for

$$\left\{ \begin{array}{l} -\Delta u + u = (v^+)^p \\ -\Delta v + v = (u^+)^q \end{array} \right. \tag{20}$$

with $p, q < (N+2)/(N-2)$. Consider

$$S = \{(u,v) \in W^{1,2}(R^N) \times W^{1,2}(R^N) : \text{ solution of (18)}\}$$

We claim that

$$\Phi_\infty = \inf\{\Phi(u,v) : (u,v) \in S \setminus \{0,0\}\}$$

is achieved. This means a ground state exists, that is a positive solution of (18) with minimum energy. Recall that

$$\Phi(u,v) = \int (\nabla u \nabla v + uv) - \int \left(\frac{|v|^{p+1}}{p+1} + \frac{|u|^{q+1}}{q+1} \right).$$

To prove the above statement it is sufficient to show that there is a constant $c > 0$ such that

$$||u||_{L^{q+1}}, ||v||_{L^{p+1}} \geq c.$$

By results of section 7 the solutions of (20) have radial symmetry. Consider only the ones that are symmetric with respect to the origin. The ones obtained by translation of these ones have the same energy. Next take a minimizing sequence. Since these functions are radial symmetric and the functional $\Phi(u,v)$ satisfies the Palais-Smale condition, we finish!

References

[BW] T. Bartsch and M. Willem, Infinitely many nonradial solutions of a Euclidean Scalar Field Equation, J. Fctl. Anal. 117(1993) p. 447-460.

[B] T.A. Benjamin, A Unified Theory of Conjugate Flows, Phil. Trans. Roy. Soc. 269(1971) p. 587-643.

[BN] H. Berestycki and L. Nirenberg, On the Method of Moving Planes and the Sliding Method, Bol. Soc. Mat. 22(1991) p. 1-37.

[BR] V. Benci and P.H. Rabinowitz, Critical Point Theorems for Indefinite Functionals, Invent. Math. 52(1979), 241-273.

[BT] H. Brézis and R. Turner, On a class of Superlinear Elliptic Problems, Comm. Part. Diff. Eq. 2 (1977), 601-614.

[C] D. C. de Morais Filho, An Ambrosetti-Prodi Type problem for an elliptic system of equations via Monotone Iteration Methods and Leray-Schauder degree theory. To appear in Abstract and Applied Analysis.

[CFM1] Ph. Clément, D.G. de Figueiredo, E. Mitidieri, Positive Solutions of Semilinear Elliptic Systems, Comm. Part. Diff. Eq. 17 (1992) p 923-940.

[CFM2] Ph. Clement, D.G. de Figueiredo, E. Mitidieri, A priori estimates for positive solutions of semilinear elliptic systems via Hardy-Sobolev inequalitie. Pitman Res. Notes in Math. vol 343, 73-91.

[CGS] L. Caffarelli, B. Gidas, J. Spruck, Assymptotic Symmetry and Local Behavior of Semilinear Elliptic Equation with critical Sobolev growth, Comm. Pure. Appl. Math. XLII (1989), 271-297.

[CL] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63(1991), 615-622.

[CM1] D.G. Costa and C.A. Magalhães, A Variational Approach to Subquadratic Perturbations of Elliptic Systems, J. Diff. Eq. 111(1994) p. 103-122.

[CM2] D.G. Costa and C.A. Magalhães, A Variational approach to Noncooperative Elliptic Systems, Nonlinear Analysis, TMA 25 (1995) p. 699-715.

[CM3] D.G. Costa and C.A. Magalhães, A Unified Approach to a Class of Strongly Indefinite Functionals, J. Diff. Eq. 122 (1996) p. 521-547.

[CS] D.G. Costa and E.A.B. Silva, in preparation.

[dF] D.G. de Figueiredo, Monotonicity and Symmetry of Solutions of Elliptic Systems in General Domains, NoDEA 1 (1994) p. 119-123.

[F] P.L. Felmer, Periodic Solutions of "superquadratic" Hamiltonean Systems, J. Diff. Eq. 102 (1993), 188-207.

[FF1] D.G. de Figueiredo and P.L. Felmer, On superquadratic elliptic systems, Trans. Amer. Math. Soc. 343 (1994) p. 99-116.

[FF2] D.G. de Figueiredo and P.L. Felmer, A Liouville-type theorem for systems, Ann. Sc. Norm. Sup. Pisa, XXI (1994) p. 387-397.

[FLN] D.G. de Figueiredo, P.L. Lions and Nussbaum, A priori estimates and existence of positive solutions of superlinear elliptic equations, J. Math. Pures Appl. 61 (1982), 41-63.

[FMa] D.G. de Figueiredo and C.A. Magalhães, On Nonquadratic Hamiltonian Elliptic Systems, Advances in Diff. Eq. , vol. 1 (1996), p. 881-898.

[FY] D.G. de Figueiredo and J. Yang, Decay, symmetry and existence of positive solutions of semilinear elliptic systems. To appear

[GNN] B. Gidas, W.M.Ni and L. Nirenberg, Symmetry and related Properties via the Maximum Principle, Comm. Math. Phys. 68 (1979) p. 209-243.

[GS1] B. Gidas and J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, Comm. Part. Dif. Eq. 6(1981), 883-901.

[GS2] B. Gidas and J. Spruck, Global and Local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34 (1981), 525-598.

[HMV] J. Hulshof, E. Mitidieri, R.C.A.M. van der Vorst, Strongly indefinite system with critical Sobolev exponents, to appear.

[HV] J. Hulshof and R.C.A.M. van der Vorst Differential Systems with Strongly Indefinite Variational Structure, J. Fctl. Anal. 114 (1993) p. 32-58.

[JQ] Jie Qing, A priori estimates for positive solutions of semilinear elliptic systems, J. Part. Dif. Eq. 1 (1988) p. 61-70.

[K] M.A. Krasnoselskii, Fixed Points of Cone-Compressing or Cone-Extending Operators, Sov. Math. Dokl. 1(1960), p. 1285-1288.

[L] P.L. Lions, The concentration-compactness principle in the Calculus of Variations, part 1, Rev. Mat. Ibero am. 1 (1985), 145-201.

[M1] E. Mitidieri - Nonexistence of positive solutions of semilinear elliptic systems in R^N. Comm. PDE

[PV] L.A. Peletier and R.C.A.M. vand der Vorst Existence and Non-existence of Positive Solutions of Non-linear Elliptic Systems and the Biharmonic Equation, Diff. Int. Eq. 5(1992) p. 747-767.

[S] M.A. Souto, Sobre a existência de soluções positivas para sistemas cooperativos não lineares, PhD thesis, UNICAMP (1992).

[St] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977) p. 149-162.

[SZ] J. Serrin and H. Zou, Non-existence of Positive Solutions of Semilinear Elliptic Systems (preprint).

[T] W.C. Troy, Symmetry Properties in Systems of Semilinear Elliptic Equations, J. Diff. Eq. 42 (1981) p. 400-413.

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