

Global hypoellipticity for $\bar{\partial}_b$ on certain compact three dimensional CR manifolds¹

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Abstract: To a smooth $(0,1)$ -form ω over a compact Riemann surface M it is possible to associate, in a natural way, a CR structure over $M \times S^1$. As a consequence of a result proved in [BCM] we derive conditions on ω that characterize the global hypoellipticity of the associated $\bar{\partial}_b$ -operator.

Key words: global hypoellipticity, CR structures, (tangential) CR operators.

Dedicated to Chaim S. Hönl on the occasion of his seventieth birthday

Introduction

The purpose of this note is to show how the results on global hypoellipticity for real operators on compact manifolds obtained in [BCM] can be used to derive analogous ones for the tangential CR operators associated to certain classes of CR manifolds. In general, if X is an abstract, locally embeddable three-dimensional CR manifold, it is well known that the associated $\bar{\partial}_b$ operator is never locally hypoelliptic (see [T1]) and it is also not globally hypoelliptic if there is some strictly pseudoconvex point in X (cf. Proposition 3 below and the argument that follows from it). Hence the study of the global hypoellipticity for such an operator has only meaning when the structure is Levi flat, in which case the validity of this property is expected to depend upon some “diophantine approximation property” related to the structure. In this work we study this problem for certain classes of CR structures defined on a product $M \times S^1$, where M is a compact Riemann surface and S^1 is the unit circle. They are naturally associated to a $(0,1)$ -form ω on M . The global hypoellipticity of the $\bar{\partial}_b$ operator is described in Theorem 1, whose proof is a consequence of the observation that it is possible to relate this property with the analogous one for $\partial_b + \bar{\partial}_b$, a real operator for which the result in [BCM] can be applied. We conclude the note by also describing the space of the global CR distributions.

Preliminaries - Statement of the main result

We start by establishing the notation to be used throughout the work. Let M be a compact Riemann surface. We will denote by $T^{1,0}$ (resp. $T^{0,1}$) the vector bundles over M whose fiber at ρ is the space of cotangent vectors of type $(1,0)$ (resp. $(0,1)$) at ρ . Thus $T^{1,0} \oplus T^{0,1} = \mathbb{C}T^*M$, the complexified cotangent bundle over M . The

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exterior derivative on M has its standard decomposition $d = \partial + \bar{\partial}$ and the space of holomorphic one-forms on M , that is, the space of $\bar{\partial}$ -closed smooth $(1,0)$ -forms on M , will be denoted by $\Omega(M)$. We will systematically identify bundles over M with bundles over $M \times S^1$ by means of their pullbacks via the natural projection $M \times S^1 \rightarrow M$.

Let θ denote the angular variable on S^1 .

Proposition 1. *Let ω be a smooth one-form on M , of type $(0,1)$, and define $\alpha = d\theta + \omega$. Then*

$$\mathbf{T}' = \mathbf{T}^{1,0} \oplus \text{span}\{\alpha\}$$

defines a rigid CR structure on $M \times S^1$.

Proof. It is clear that \mathbf{T}' defines a subbundle of $\mathbb{C}\mathbf{T}^*(M \times S^1)$. Furthermore we have $\mathbf{T}' + \overline{\mathbf{T}'} = \mathbb{C}\mathbf{T}^*(M \times S^1)$ for $d\theta = \alpha - \omega$ is a section of $\mathbf{T}' + \overline{\mathbf{T}'}$. Let \mathcal{V} denote the orthogonal (for the duality between one-forms and vector fields) of \mathbf{T}' . It follows immediately that \mathcal{V} is a complex line bundle over $M \times S^1$ satisfying

$$\mathcal{V} \cap \overline{\mathcal{V}} = 0, \quad \mathbb{C}\mathbf{T}(M \times S^1) = \mathcal{V} \oplus \overline{\mathcal{V}} \oplus \text{span}\left\{\frac{\partial}{\partial\theta}\right\}, \quad \left[\frac{\partial}{\partial\theta}, \mathcal{V}\right] = 0.$$

The proof is complete.

The tangential CR operator associated to such a structure is the first order operator

$$\bar{\partial}_b : C^\infty(M \times S^1) \longrightarrow C^\infty(M \times S^1; \mathbb{C}\mathbf{T}^*M)$$

given by

$$\bar{\partial}_b u = \bar{\partial}u - \omega \frac{\partial u}{\partial\theta}.$$

The main goal of the present note is to derive conditions in order to ensure that $\bar{\partial}_b$ is globally hypoelliptic in the sense of the following definition:

Definition 1. *Let E and F be vector bundles over a smooth manifold \mathcal{M} and let $P : C^\infty(\mathcal{M}, E) \rightarrow C^\infty(\mathcal{M}, F)$ be a linear partial differential operator acting between sections of E and sections of F . The operator P is said to be globally hypoelliptic if the conditions $u \in \mathcal{D}'(\mathcal{M}, E)$, $Pu \in C^\infty(\mathcal{M}, F)$ imply $u \in C^\infty(\mathcal{M}, E)$.*

In order to state our main result some preparation is necessary. We begin by applying Theorem 19.9 in [F]. We can write, in a unique way,

$$(1) \quad \omega = \bar{\vartheta} + \bar{\partial}G$$

where $G \in C^\infty(M)$ and $\vartheta \in \Omega(M)$. In local coordinates we have

$$\omega = \left(\frac{\partial G}{\partial \bar{z}} + \bar{\lambda} \right) d\bar{z}$$

where λ is holomorphic. Thus \mathcal{V} is locally spanned by the vector field

$$L = \frac{\partial}{\partial \bar{z}} - \left(\frac{\partial G}{\partial \bar{z}} + \bar{\lambda} \right) \frac{\partial}{\partial \theta}$$

and a simple computation gives

$$[L, \bar{L}] = 2i \frac{\partial \text{Im}G}{\partial z \partial \bar{z}} \frac{\partial}{\partial \theta}.$$

Thus \mathbf{T}' is strictly pseudoconvex at every point $(z, \theta) \in M \times S^1$ such that

$$\frac{\partial \text{Im}G}{\partial z \partial \bar{z}}(z) \neq 0$$

and Levi flat if and only if $\text{Im}G$ is constant. It is important to point out that such a structure can never be strictly pseudoconvex everywhere due to the maximum principle for subharmonic functions.

Proposition 2. *The following properties are equivalent:*

- (a) \mathbf{T}' is Levi flat;
- (b) $d(\omega + \bar{\omega}) = 0$.

Proof. We first show that (a) \Rightarrow (b). According to the preceding discussion, in the decomposition (1) the function G can be assumed real. Thus

$$\omega + \bar{\omega} = \bar{\vartheta} + \vartheta + \bar{\partial}G + \partial G = \bar{\vartheta} + \vartheta + dG$$

and hence

$$d(\omega + \bar{\omega}) = \partial\bar{\vartheta} + \bar{\partial}\vartheta = 0.$$

Now we assume that (b) holds. As above we get

$$d(\bar{\partial}G + \partial\bar{G}) = 0.$$

On the other hand

$$d\bar{\partial}G + d\partial\bar{G} = \partial\bar{\partial}(G - \bar{G}) = 2i\partial\bar{\partial}(\text{Im}G);$$

hence $\text{Im}G$ is harmonic and thus constant. The proof is complete.

Finally we recall the following definition (see [BCM]):

Definition 1. *For a closed, smooth and real one-form a on M , we define:*

- (a) a is **integral** if $\frac{1}{2\pi} \int_{\sigma} a \in \mathbb{Z}$ for any one-cycle σ in M .
- (b) a is **rational** if there exists $q \in \mathbb{N}$ such that qa is an integral one-form.
- (c) a is **Liouville** if a is not rational and there exist a sequence of closed, integral one-forms $\{a_j\}$ and a sequence of integers $q_j \geq 2$ such that $\left\{ q_j^j \left(a - \left(\frac{1}{q_j} \right) a_j \right) \right\}$ is bounded in $C^\infty(M, \mathbb{C}T^*M)$.

It is easily seen that a is integral if and only if its cohomology class $[a] \in H^1(M, \mathbb{R})$ belongs to the image of the natural homomorphism

$$H^1(M, 2\pi\mathbb{Z}) \rightarrow H^1(M, \mathbb{R}).$$

This remark shows that the terminology here adopted coincides (up to a factor 2π) with the classical one (see, e.g., [LB, p.36]).

We are now in position to state our main result:

Theorem 1. *The following conditions are equivalent:*

- (a) *The operator $\bar{\partial}_b$ is globally hypoelliptic;*
- (b) *T' is Levi flat and $\omega + \bar{\omega}$ is neither rational nor Liouville.*

Proof of Theorem 1

We begin with a simple (and well known) observation:

Proposition 3. *Let \mathcal{T} be a locally integrable CR structure on an open subset V of \mathbb{R}^3 . Assume that \mathcal{T} is strictly pseudoconvex at $\rho_0 \in V$. Then, near ρ_0 , there is a continuous CR function whose singular support equals $\{\rho_0\}$.*

Proof. The proof is absolutely routine. It is well known (see, e.g., [T2, sections I.7 and I.9]) that there are coordinates (x, y, s) centered at ρ_0 such that \mathcal{T} is spanned by dz, dw , where $w = s + i\gamma(z, s)$. Here $z = x + iy$ and γ is real, smooth and satisfies

$$\gamma(z, s) = |z|^2 + O(|z|^3 + |z||s| + s^2).$$

In a small neighborhood of the origin we have $\gamma(z, 0) \geq |z|^2/2$ and thus $v(z, s) = \gamma(z, s) - is$ has image contained in $\mathbb{C} \setminus (]-\infty, 0[+ i\{0\})$ and vanishes if and only if $z = 0$ and $s = 0$. Taking \sqrt{v} , with the choice of the main branch of the square root, gives the desired CR function.

We now begin the proof of Theorem 1. It is a consequence of Proposition 3 that if T' is not Levi flat then $\bar{\partial}_b$ cannot be globally hypoelliptic. In fact, if there is a point $\rho \in M \times S^1$ where the structure is strictly pseudoconvex then, by Proposition 3, we get the existence of an open neighborhood U_0 of ρ in $M \times S^1$ and of $u \in C^0(U_0)$ such that

$$\bar{\partial}_b u = 0, \quad \text{singsupp } u = \{\rho\}.$$

If take $\chi \in C_c^\infty(U_0)$, with $\chi \equiv 1$ near ρ then $\chi u \in C^0(M \times S^1)$, $\bar{\partial}_b(\chi u) \in C^\infty(M \times S^1, \mathbb{C}T^*M)$ and $\text{singsupp}(\chi u) = \{\rho\}$. This proves our assertion.

We now introduce the following real operator

$$\mathbb{L} : C^\infty(M \times S^1) \rightarrow C^\infty(M \times S^1, \mathbb{C}T^*M)$$

defined by

$$\mathbb{L}u = du - (\omega + \bar{\omega}) \frac{\partial u}{\partial \theta}.$$

Its relevance to our study relies on the following fact: according to [BCM, Theorem 2.4], if $\omega + \bar{\omega}$ is closed then the operator \mathbb{L} is globally hypoelliptic if and only if $\omega + \bar{\omega}$ is neither rational nor Liouville. Taking Proposition 2 into account it follows that Theorem 1 will be a consequence of the following result:

Proposition 4. *Suppose that $\omega + \bar{\omega}$ is closed. If $u \in \mathcal{D}'(M \times S^1)$ then $\bar{\partial}_b u \in C^\infty(M \times S^1, \mathbb{C}T^*M)$ if and only if $\mathbb{L}u \in C^\infty(M \times S^1, \mathbb{C}T^*M)$. In particular the operator $\bar{\partial}_b$ is globally hypoelliptic if and only if the same is true for the operator \mathbb{L} .*

Proof. We select a hermitian metric on M . This metric defines hermitian inner products on each fiber of $\mathbb{C}T^*M$ and of $\Lambda^2\mathbb{C}T^*M$ which, in a local holomorphic coordinate $z = x + iy$, satisfy

$$\begin{aligned} \langle dz, dz \rangle &= \langle d\bar{z}, d\bar{z} \rangle = h^{-1}, & \langle dz, d\bar{z} \rangle &= 0; \\ \langle dz \wedge d\bar{z}, dz \wedge d\bar{z} \rangle &= h^{-2}, \end{aligned}$$

for some $h > 0$ [LB, p.11]. Moreover, the corresponding volume element has the expression $dV = 2hdxdy$. We can thus consider the Hilbert spaces $L^2(M; dV)$, $L^2(M \times S^1; dVd\theta)$, as well as the Hilbert space $L^2(M, \mathbb{C}T^*M; dV)$ (resp. $L^2(M \times S^1, \mathbb{C}T^*M; dVd\theta)$) of measurable sections f of $\mathbb{C}T^*M$ over M (resp. $M \times S^1$) such that

$$\int_M \langle f, f \rangle dV < \infty \quad (\text{resp. } \int_{M \times S^1} \langle f, f \rangle dVd\theta < \infty).$$

Taking adjoints with respect to the corresponding hilbertian structures allows us to introduce the operators

$$\bar{\partial}_b^* \bar{\partial}_b, \mathbb{L}^* \mathbb{L} : C^\infty(M \times S^1) \rightarrow C^\infty(M \times S^1).$$

We now compute them explicitly.

Lemma 1. *Let $\omega \in C^\infty(M, T^{0,1})$. Then:*

- The adjoint of $\omega \frac{\partial}{\partial \bar{\theta}} : C^\infty(M \times S^1) \rightarrow C^\infty(M \times S^1, \mathbb{C}T^*M)$ is the map $\beta \mapsto - \langle \frac{\partial \beta}{\partial \bar{\theta}}, \omega \rangle$.
- $\bar{\partial}^*(u\omega) = u\bar{\partial}^*\omega - \langle \partial u, \bar{\omega} \rangle$ for every $u \in C^\infty(M \times S^1)$.
- $\langle du, \omega + \bar{\omega} \rangle = \langle \partial u, \bar{\omega} \rangle + \langle \bar{\partial} u, \omega \rangle$ for every $u \in C^\infty(M \times S^1)$.
- $|\omega + \bar{\omega}|^2 = 2|\omega|^2$.
- If $d(\omega + \bar{\omega}) = 0$ then $d^*(\omega + \bar{\omega}) = 2\bar{\partial}^*\omega$.

Proof. All the properties are of easy verification. We content ourselves in proving property (e). In a local holomorphic coordinate $z = x + iy$ we have

$$\bar{\partial}^*(Bd\bar{z}) = -h^{-1} \frac{\partial B}{\partial z}, \quad d^*(b_1 dx + b_2 dy) = -(2h)^{-1} \left(\frac{\partial b_1}{\partial x} + \frac{\partial b_2}{\partial y} \right).$$

Now write $\omega = \text{Ad}\bar{z}$. Then $\omega + \bar{\omega} = 2(\text{Re}A dx + \text{Im}A dy)$ and consequently

$$\begin{aligned}\bar{\partial}^* \omega &= -h^{-1} \frac{\partial A}{\partial z} \\ &= -(2h)^{-1} \left(\frac{\partial \text{Re}A}{\partial x} + i \frac{\partial \text{Im}A}{\partial x} - i \frac{\partial \text{Re}A}{\partial y} + \frac{\partial \text{Im}A}{\partial y} \right) \\ &= \frac{1}{2} d^*(\omega + \bar{\omega})\end{aligned}$$

since $d(\omega + \bar{\omega}) = 0$. The proof is complete.

From (a) of Lemma 1 we obtain

$$\bar{\partial}_b^* = \bar{\partial}^* + \left\langle \frac{\partial}{\partial \theta}, \omega \right\rangle$$

and, consequently, (b) of Lemma 1 gives

$$(2) \quad \bar{\partial}_b^* \bar{\partial}_b = \bar{\partial}^* \bar{\partial} + \left\langle \frac{\partial}{\partial \theta}, \bar{\partial}, \omega \right\rangle + \left\langle \frac{\partial}{\partial \theta}, \bar{\partial}, \bar{\omega} \right\rangle - |\omega|^2 \frac{\partial^2}{\partial \theta^2} - (\bar{\partial}^* \omega) \frac{\partial}{\partial \theta}.$$

By the same argument

$$(3) \quad \mathbb{L}^* \mathbb{L} = d^* d + 2 \left\langle \frac{\partial}{\partial \theta}, d, \omega + \bar{\omega} \right\rangle - |\omega + \bar{\omega}|^2 \frac{\partial^2}{\partial \theta^2} - (d^*(\omega + \bar{\omega})) \frac{\partial}{\partial \theta}$$

and thus, (c), (d) and (e) of Lemma 1 in conjunction with the fact that $d^* d = 2\bar{\partial}^* \bar{\partial}$ imply

$$(4) \quad \mathbb{L}^* \mathbb{L} = 2\bar{\partial}_b^* \bar{\partial}_b.$$

This identity is the key tool for the proof of Proposition 4. Let then $u \in \mathcal{D}'(M \times S^1)$ satisfy $\bar{\partial}_b u \in C^\infty(M \times S^1, \mathbb{C}T^*M)$. By standard results on partial hypoellipticity (see, e.g., [T2, Section I.4]) u must necessarily belong to $C^\infty(M, \mathcal{D}'(S^1))$ and thus its Fourier expansion in θ can be written as

$$u = \sum_{j \in \mathbb{Z}} u_j e^{ij\theta}$$

where $u_j \in C^\infty(M)$. We have

$$(5) \quad \bar{\partial}_b u = \sum_{j \in \mathbb{Z}} (\bar{\partial} u_j - ij\omega u_j) e^{ij\theta},$$

$$(6) \quad \mathbb{L}u = \sum_{j \in \mathbb{Z}} (du_j - ij(\omega + \bar{\omega})u_j) e^{ij\theta}.$$

Now if we take adjoints with respect to the hilbertian structures of $L^2(M; dV)$ and $L^2(M, \mathbb{C}T^*M; dV)$ it follows immediately from (4) that

$$2(\bar{\partial} - ij\omega)^*(\bar{\partial} - ij\omega) = (d - ij(\omega + \bar{\omega}))^* (d - ij(\omega + \bar{\omega})).$$

Hence

$$(7) \quad \|(\bar{\partial} - ij\omega)u_j\|^2 = \frac{1}{2} \| (d - ij(\omega + \bar{\omega})) u_j \|^2$$

with norms in $L^2(M, \mathbb{C}T^*M; dV)$. Summing up, (5), (6) and (7) show that the following property holds true:

$$(8) \text{ If } u \in \mathcal{D}'(M \times S^1) \text{ and } \bar{\partial}_b u \in C^\infty(M \times S^1, \mathbb{C}T^*M) \text{ then } \mathbb{L}u \in L^2(M \times S^1, \mathbb{C}T^*M; dVd\theta).$$

Let again $u \in \mathcal{D}'(M \times S^1)$ be such that $\bar{\partial}_b u \in C^\infty(M \times S^1, \mathbb{C}T^*M)$. If μ is an arbitrary parameter we have

$$\bar{\partial}_b \left(\bar{\partial}_b^* \bar{\partial}_b u - \mu \frac{\partial^2 u}{\partial \theta^2} \right) = \left(\bar{\partial}_b \bar{\partial}_b^* - \mu \frac{\partial^2}{\partial \theta^2} \right) \bar{\partial}_b u \in C^\infty(M \times S^1, \mathbb{C}T^*M)$$

and thus from (8) we obtain

$$\mathbb{L} \left(\bar{\partial}_b^* \bar{\partial}_b u - \mu \frac{\partial^2 u}{\partial \theta^2} \right) \in L^2(M \times S^1, \mathbb{C}T^*M; dVd\theta).$$

Taking (4) into account this last property can be rewritten as

$$(9) \quad \mathbb{L} \left(\frac{1}{2} \mathbb{L}^* \mathbb{L} u - \mu \frac{\partial^2 u}{\partial \theta^2} \right) \in L^2(M \times S^1, \mathbb{C}T^*M; dVd\theta).$$

Finally we introduce the operator

$$\mathbb{L}_\bullet : C^\infty(M \times S^1, \mathbb{C}T^*M) \longrightarrow C^\infty(M \times S^1, \Lambda^2 \mathbb{C}T^*M)$$

defined by

$$\mathbb{L}_\bullet f = df - (\omega + \bar{\omega}) \wedge \frac{\partial f}{\partial \theta}.$$

Since $\omega + \bar{\omega}$ is closed it follows that $\mathbb{L}_\bullet \mathbb{L} = 0$ and thus (9) gives

$$(10) \quad \frac{1}{2} \left(\mathbb{L}_\bullet^* \mathbb{L}_\bullet + \mathbb{L} \mathbb{L}^* - 2\mu \frac{\partial^2}{\partial \theta^2} \right) \mathbb{L}u \in L^2(M \times S^1, \mathbb{C}T^*M; dVd\theta).$$

Now, it is easily seen that $\mathbb{L}_\bullet^* \mathbb{L}_\bullet + \mathbb{L} \mathbb{L}^* - 2\mu \frac{\partial^2}{\partial \theta^2}$ is elliptic if $\mu > 0$ is chosen large enough and consequently (10) implies that $\mathbb{L}u$ belongs to $H^2(M \times S^1, \mathbb{C}T^*M)$, the Sobolev space of order 2. In conclusion we have improved property (8) in the sense that $L^2(M \times S^1, \mathbb{C}T^*M; dVd\theta)$ can be replaced by $H^2(M \times S^1, \mathbb{C}T^*M)$

in its statement. By iterating the argument we reach the conclusion that $L^2(M \times S^1, \mathbb{C}T^*M; dVd\theta)$ can in fact be replaced by $\cap_{k>0} H^k(M \times S^1, \mathbb{C}T^*M) = C^\infty(M \times S^1, \mathbb{C}T^*M)$ in property (8). To complete the proof of Proposition 4 it suffices then to notice that we can interchange the roles of $\bar{\partial}_b$ and \mathbb{L} in the whole argument.

Final Remark: Global CR distributions

For these CR structures the kernel of the operator $\bar{\partial}_b$ can be completely described. We keep the notation established throughout the work and take $u \in \mathcal{D}'(M \times S^1)$ satisfying $\bar{\partial}_b u = 0$. Then $u = \sum_{j \in \mathbb{Z}} u_j e^{ij\theta}$, where each u_j is smooth on M and satisfies

$$(11) \quad (\bar{\partial} - ij\omega)u_j = 0.$$

Let $p: \tilde{M} \rightarrow M$ be the universal covering of M . For the form ϑ introduced in (1) we can find a holomorphic function h on \tilde{M} such that $\partial h = p^*\vartheta$. Then

$$\bar{\partial}(h + \bar{h} + G \circ p) = p^*\omega$$

and hence (11) implies that

$$H_j \doteq e^{-ij(h + \bar{h} + G \circ p)}(u_j \circ p)$$

is a holomorphic function on \tilde{M} . Now, since $|H_j| = \exp\{j \operatorname{Im} G \circ p\} |u_j \circ p|$ is a well defined subharmonic function on M it follows that $|H_j|$ is constant. Consequently H_j is itself a constant and we can write

$$(12) \quad u_j \circ p = c_j e^{ij(h + \bar{h} + G \circ p)}$$

with $c_j \in \mathbb{C}$.

Suppose first that $\vartheta + \bar{\vartheta}$ is not a rational form. If $j \in \mathbb{Z}$, $j \neq 0$ then there are points A and B in \tilde{M} such that $p(A) = p(B)$ and

$$j\{(h + \bar{h})(A) - (h + \bar{h})(B)\} \notin 2\pi\mathbb{Z}.$$

From the relation

$$(e^{-ijG} u_j) \circ p = c_j e^{ij(h + \bar{h})}$$

it then follows that $c_j = 0$. Suppose otherwise that $\vartheta + \bar{\vartheta}$ is rational and let ℓ be the smallest natural number such that $\ell(\vartheta + \bar{\vartheta})$ is integral. Then $e^{i\ell(h + \bar{h})}$ is a well defined smooth function on M and the same argument as above shows that $c_j = 0$ if $j \notin \ell\mathbb{Z}$. We summarize this discussion as follows (cf. also [BCP, Lemma 2.2]):

Proposition 5. *If $\vartheta + \bar{\vartheta}$ is not a rational form then the kernel of $\bar{\partial}_b$ in $\mathcal{D}'(M \times S^1)$ contains only the constants. If otherwise $\vartheta + \bar{\vartheta}$ is a rational form then the kernel of $\bar{\partial}_b$ in $\mathcal{D}'(M \times S^1)$ is the space of all distributions of the form*

$$u = \sum_{j \in \mathbb{Z}} a_j e^{ij\ell(\theta + h + \bar{h} + G)},$$

where $a_j \in \mathbb{C}$ and h, ℓ , and G have the meaning explained above. In particular, when $\vartheta + \bar{\vartheta}$ is rational, all the global CR functions are functionally dependent on a single generating function.

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